On stable reconstructions from univariate nonuniform Fourier measurements

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Abstract

In this paper, we consider the fundamental problem of recovering a univariate function f from a finite collection of pointwise samples of its Fourier transform taking nonuniformly. In the first part of the paper, we show that, under suitable conditions on the sampling frequencies – specifically, its density and its bandwidth – it is possible to recover f in a stable and accurate manner in any given finite-dimensional approximation subspace. In practice, this can be carried out using so-called nonuniform generalized sampling (NUGS). In the second part of the paper we consider approximation spaces consisting of compactly supported wavelets. We prove that a linear scaling of the dimension of the space with the sampling bandwidth is both necessary and sufficient for stable and accurate recovery. Thus wavelets are up to constant factors optimal spaces for reconstruction.

1 Introduction

The reconstruction of an image or signal from a collection of its Fourier measurements is an important task in applied mathematics. This problem arises in numerous applications, ranging from medical imaging (e.g. Magnetic Resonance Imaging (MRI)) to X-ray tomography, seismology and microscopy (the latter processes usually involve the Radon transform, but equate to reconstruction from Fourier measurements via the Fourier slice theorem).

The purpose of this paper is to consider the following fundamental question: given fixed measurements of an unknown image f, i.e. a finite collection of samples of its Fourier transform $\hat{f}$, under what conditions is possible to recover (an approximation to) f, and how can this be achieved with a stable numerical algorithm? Our main contributions are: (i) a comprehensive theoretical framework in the univariate setting which explains when stable reconstruction is possible (ii) a stable numerical algorithm to achieve such reconstructions and (iii) analysis for the important case of reconstructions in wavelet bases.

The particular focus of this paper is on the case where the data is acquired nonuniformly in the Fourier domain. Nonuniform sampling arises naturally in many of the applications listed above. In particular, radial sampling of the Fourier transform results whenever sampling with the Radon transform. Furthermore, nonuniform sampling patterns – in particular, spiral trajectories (see [9, 17] and references therein) – have become increasingly popular amongst MRI practitioners in the last several decades. Due primarily to the physics of the MR scanner, these sampling geometries allow for faster, uninterrupted scanning over a larger range of frequencies than traditional Cartesian scans. This results in higher resolution in Fourier space with similar acquisition times, and in theory, reconstructions from such sampling geometries correspondingly possess fewer errors in comparison to Cartesian sampling, due to both the higher resolution obtained and the lower magnetic gradients required to scan along such trajectories [24, 28, 31].

1.1 Generalized sampling

The approach we take in this paper is based on recent developments in sampling and reconstruction in abstract Hilbert spaces, known as *generalized sampling (GS)*. This general framework, introduced in [2] and based on previous work of Unser & Aldroubi [39] and later Eldar et al. [18, 19], addresses the following problem in sampling theory. Suppose that samples of an element f of a Hilbert space are given as inner products with respect to a particular basis or frame. Moreover, suppose it is known that f can be efficiently represented in another basis or frame (e.g. it has sparse or rapidly-decaying coefficients). GS obtains a reconstruction of f in this new system using only the original sampling data. In the linear case, this is achieved by least-squares fitting [3], but when sparsity is assumed, one can combine it with compressed sensing techniques (i.e. convex optimization and random sampling) to achieve substantial subsampling [1]. By doing so, one obtains a theory and set of techniques for infinite-dimensional (i.e. analog) compressed sensing, referred to as GS–CS.

GS, or GS–CS, naturally applies to the Fourier reconstruction problem whenever the samples are taken uniformly. The primary advantage of GS over other standard reconstruction algorithms for this problem (e.g. gridding or resampling – see below) is that it allows one to take advantage of the availability of efficient representation systems for images. It is well known that natural images are well represented in wavelet systems. Images may be sparse in wavelets, or have coefficients with rapid decay. Moreover, representing medical images in such systems has other benefits over classical Fourier series representations, such as improved compressibility, better feature detection and easier and more effective denoising [29, 32, 41]. GS allows one to compute quasi-optimal reconstructions in wavelets from the given set of Fourier samples, and therefore exploit such beneficial properties. In the case of uniform Fourier samples, the use of GS with wavelets was extensively studied in [5]. See also [6] for the case of GS–CS with wavelets.

1.2 Contributions of the paper and relation to previous work

The purpose of this paper is to extend the GS framework to the case where Fourier samples are acquired nonuniformly. We refer to the resulting framework as *nonuniform generalized sampling*

(NUGS). Specifically, suppose that $\Omega = \{\omega_1, \ldots, \omega_N\} \subseteq \mathbb{R}$ is a set of N frequencies, and that we are given the measurements

$$\{\hat{f}(\omega) : \omega \in \Omega\},$$

of an unknown signal $f \in \mathrm{L}^2(0, 1)$. Note that for the purposes of this paper, Ω is fixed and cannot be readily altered (this is typical in the applications listed above). Let $\mathrm{T} \subseteq \mathrm{L}^2(0, 1)$ be a finite-dimensional space in which we wish to recover f. For example, T could consist of the first M functions in some orthonormal wavelet basis. Our main contribution is to derive conditions on Ω and T under which stable reconstruction is possible with NUGS. In particular, we show that if the samples Ω have *density* $\delta < 1$ then stable reconstruction is possible provided the *bandwidth* K of Ω is sufficiently large, with the precise nature of the scaling depending on the properties of T. We shall also address the case of the critical density $\delta = 1$ within the setting of Fourier frames.

An important facet of NUGS is that it is always possible to compute the various constants that enter into our stability and error estimates. Thus, for a given Ω and T, stable reconstruction can be guaranteed *a priori* by a straightforward numerical calculation. Our numerical results illustrate that these bounds give good estimates of the actual reconstruction errors seen in practice.

The first part of this paper, §2–4, is devoted to the general theory of NUGS in the univariate setting. In the second half, we focus on the important case where the subspace T corresponds to a wavelet basis. A result proved in [5] shows that when the sampling set Ω consists of the first N uniform frequencies one can recover the first $\mathcal{O}(N)$ coefficients in an arbitrary wavelet basis using GS. Thus wavelet bases are, up to constants, optimal bases in which to recover images from uniform Fourier samples. This is not true for example with algebraic polynomial bases, in which case one can stably recover only at best the first $\mathcal{O}(\sqrt{N})$ coefficients [3]. One of the main contributions of the second half of this paper is to extend this result to the nonuniform case. Specifically, if the samples Ω have bandwidth K and density $\delta < 1$ then we prove that one can recover the first $\mathcal{O}(K)$ wavelet coefficients using NUGS. Thus, perhaps unsurprisingly, there is a fundamental one-to-one relationship between the sampling bandwidth and the wavelet scale. This is further highlighted in §6 where we establish a result that shows that any attempt to reconstruct a fixed number of wavelet coefficients from a sampling bandwidth K below a critical threshold necessarily results in exponential ill-conditioning. This generalizes a result proved for GS in [5] to the nonuniform setting and is related to recent work of Candès & Fernandez–Granda on super-resolution [13].

Let us now make several further remarks. For the remainder of this paper we shall focus on the case of one-dimensional functions f having compact support in $[0, 1]$. It is, however, possible to extend our results to the higher-dimensional setting, and this is currently work in progress. Another issue we shall not address in this paper is that of sparsity. Sparsity-exploiting algorithms are currently revolutionising signal and image reconstruction. Since a main focus of this paper is wavelets, in which images are known to be sparse, it may at first sight appear strange not to seek to exploit such properties. For uniform samples this has indeed been done by using the aforementioned GS–CS framework, and the results are reported in [1, 6]. However, as was explained in [1] (see also [7]), before one can exploit sparsity it is first necessary to understand the fundamental linear mapping between the samples and coefficients in the reconstruction system. This is precisely what we do in this paper via NUGS. Exploiting sparsity by extending the work of [6] to the case of fully nonuniform Fourier samples is a topic of ongoing investigations.

An important facet of this work is that we assume an analog model for the unknown image or signal f. Unlike other common approaches in nonuniform sampling, such as gridding [26, 36, 40], resampling [35, 34] or iterative reconstructions [38], f is not modelled as a finite-length Fourier series, or as a finite array of pixels, but rather as a function in $\mathrm{L}^2(0, 1)$. Consequently, a key issue in NUGS is that of *approximation*. By using an appropriate approximation basis for the function, we successfully avoid the unpleasant artefacts (e.g. Gibbs ringing) associated with gridding and resampling algorithms. Note that the popular iterative reconstruction algorithm of Sutton, Noll and Fessler [38] is a special case of our framework NUGS corresponding to a pixel basis for T. Thus our work provides as a corollary important theoretical guarantees for the stability and error of this algorithm that, to the best of our knowledge, are currently unknown.

In addition, we note that, unlike in the standard compressed sensing approach to MRI, we model the measurements as continuous Fourier samples rather than discrete Fourier samples. As explained in [1, 7, 25], modelling the measurements as discrete Fourier samples leads to basis mismatch and an inverse crime, and consequently results in inferior reconstructions. Note that our analog model is the same as that used with great success in recent work of Guerquin-Kern, Haberlin, Pruessmann & Unser [24] on iterative, wavelet-based reconstruction algorithms for MRI. The results we prove in this paper mark the first step towards theoretical reconstruction guarantees for these algorithms.

Finally, we remark that it is common in nonuniform sampling theory to assume that the sampling frequencies give rise to a Fourier frame [10, 11, 12, 20]. Reconstruction can then be carried out by iterative inversion of the frame operator, for example (this reconstruction is quite different from ours, though, since the approximation properties are tied to those of the sampling frame). However, the frame assumption can be problematic for several reasons. First, Fourier frames do not allow arbitrary clustering of sampling frequencies, such as often the case in practice. Second, even if a sequence of samples gives rise to a frame, it can be difficult to determine the frame constants so as to get explicit bounds. Moreover, in our setting, where we consider finite sets of samples, and reconstructions in finite-dimensional spaces, the existence of a countable frame sequence is in some senses superfluous. Instead, we shall mainly focus on simple conditions for stable recovery – namely, the density δ and the bandwidth K – both of which can be easily computed, and importantly, generalize to higher dimensions (this will be reported in a subsequent work). Clustering of the samples will be accounted for by appropriate weighting, which is similar to the approach taken in [21, 22, 33]. One may consider this as a weighted Fourier frame approach, although we do not necessarily assume an infinite number of samples. Having said this, however, we will also provide theorems for NUGS for the case of sampling with Fourier frames. These results are important, since they allow one to analyse the reconstruction when sampling at the critical density $\delta = 1$.

2 The reconstruction problem

Let us first introduce some notation. Throughout the paper, $\langle \cdot, \cdot \rangle$ will denote the inner product on $\mathrm{L}^2(\mathbb{R})$ and $\|\cdot\|$ the corresponding norm. We denote the Fourier transform by

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x) \mathrm{e}^{-2\pi \mathrm{i} \omega x} \,\mathrm{d}x, \quad \omega \in \mathbb{R}, \quad f \in \mathrm{L}^2(\mathbb{R}).$$

Our primary consideration in this paper will be functions with compact support, which we normalize to $[0,1]$. Thus, we define the subspace

$$\mathrm{H} = \left\{ f \in \mathrm{L}^2(\mathbb{R}) : \ \mathrm{supp}(f) \subseteq [0,1] \right\} \subseteq \mathrm{L}^2(\mathbb{R}).$$

We shall use the notation

$$\Omega = \{\omega_1, \ldots, \omega_N\} \subseteq \mathbb{R},$$

to denote a finite set of distinct frequencies. For convenience we assume that these points are ordered with $\omega_1 < \ldots < \omega_N$. We shall refer to such a set Ω as a *sampling scheme*. We also define T to be a finite-dimensional subspace of H; the so-called *reconstruction space*. The corresponding orthogonal projection onto T is denoted by $\mathcal{P}_{\mathrm{T}}$.

Given such a sampling scheme Ω and reconstruction space $\mathrm{T} \subseteq \mathrm{H}$, the reconstruction problem we address in this paper is that of computing an approximation $\tilde{f} \in \mathrm{T}$ to f in the subspace T using only the sampling data

$$\{\hat{f}(\omega) : \omega \in \Omega\}. \tag{2.1}$$

When developing a method for this problem, i.e. a mapping F from f to $\tilde{f}$ depending solely on the data (2.1), there are two critical considerations (see [4] for further discussion):

(i) The mapping F should be *quasi-optimal*. That is, for some constant $\mu = \mu(F) \ll \infty$ independent of f, we have
$$\|f - F(f)\| \leq \mu \|f - \mathcal{P}_{\mathrm{T}} f\|, \quad \forall f \in \mathrm{H}.$$

(ii) The mapping F should be *numerically stable*, i.e. or some constant $\kappa = \kappa(F) \ll \infty$,
$$\|F(g)\| \leq \kappa \|g\|, \quad \forall g \in \mathrm{H}.$$

Note that a quasi-optimal mapping F is automatically numerically stable with constant $\kappa \leq 1 + \mu$. We include both mainly for emphasis, and also because later we will derive exact expressions for both κ and μ in the case of NUGS. Note also that any quasi-optimal mapping F satisfies

$$\|f - F(f+g)\| \leq C \left(\|f - \mathcal{P}_{\mathrm{T}} f\| + \|g\| \right), \quad \forall f, g \in \mathrm{H},$$

where $C \leq 1 + \mu$ in general and $C = \max\{\kappa, \mu\}$ in the case of linear F (such as NUGS).

Recall that the motivation for considering a particular reconstruction space T, e.g. the span of the first M wavelets, is that f is known to be well-represented in this space. In other words, the error $\|f - \mathcal{P}_{\mathrm{T}} f\|$ is small. Quasi-optimality guarantees that the reconstruction $\tilde{f}$ from the data (2.1) inherits such a small error. Stability, on the other hand, is also vital to ensure that perturbations of the measurements do not adversely affect the reconstruction.

With this, we can now formalize the main problem we address in this paper:

Problem 2.1 (The reconstruction problem)**.** *Given a sampling scheme Ω and a reconstruction space T, determine:*

(i) under what conditions stable, quasi-optimal reconstruction is possible,
(ii) the magnitude of the corresponding reconstruction constant $C = C(\Omega, \mathrm{T})$.

We shall solve this problem by analysing a particular method, the NUGS reconstruction, which we introduce in §3. This provides a sufficient condition for (i) and an upper bound for (ii). As we explain in Remark 6.4, however, under appropriate conditions the NUGS reconstruction cannot be outperformed by any other method. In other words, the NUGS reconstruction constant is a fundamental constant of the mapping between samples (2.1) and the subspace T. Hence our analysis of NUGS provides not only sufficient conditions for stable, quasi-optimal reconstruction, but also (under appropriate, but mild, assumptions) necessary conditions.

3 Nonuniform generalized sampling

3.1 The case of uniform samples

Problem 2.1 was originally considered in [2, 3] for the case of uniform samples. If N is odd this corresponds to setting

$$\Omega = \{\epsilon n : n \in \mathbb{Z} \cap [-(N-1)/2, (N-1)/2]\}, \tag{3.1}$$

where $\epsilon \leq 1$ is the sampling period ($\epsilon = 1$ corresponds to the Nyquist rate). The reconstruction $\tilde{f}$ is then defined by a least-squares fit of the data:

$$\tilde{f} = \underset{g \in \mathrm{T}}{\operatorname{argmin}} \sum_{n=-(N-1)/2}^{(N-1)/2} \left| \hat{f}(\epsilon n) - \hat{g}(\epsilon n) \right|^2. \tag{3.2}$$

As was shown in [2], whenever N is sufficiently large (with the precise value depending on T), then the mapping $F : f \mapsto \tilde{f}$ given by (3.2) is stable and quasi-optimal.

Before extending GS to the case of nonuniform samples, let us first explain why letting $N \to \infty$ leads to a bounded reconstruction constant. First, we note that if

$$\mathcal{S}f(x) = \frac{1}{\epsilon} \sum_{n=-(N-1)/2}^{(N-1)/2} \hat{f}(\epsilon n) \mathrm{e}^{2\pi \mathrm{i} \epsilon n x} \mathbb{I}_{[0,1]}(x), \tag{3.3}$$

denotes the partial Fourier series of a function f, then one can show the following: (3.2) is equivalent to the variational problem

$$\text{find } \tilde{f} \in \mathrm{T} \text{ such that } \langle \mathcal{S}\tilde{f}, g \rangle = \langle \mathcal{S}f, g \rangle, \ \forall g \in \mathrm{T}.$$

The operator $\mathcal{S}$, the partial Fourier operator, converges strongly to the identity operator on H as $N \to \infty$. Thus, for large N, $\langle \mathcal{S}\cdot, \cdot \rangle$ defines an equivalent inner product over the finite-dimensional subspace T. Stability and quasi-optimality can now be deduced using similar arguments to those that we present in the case of NUGS in §3.3. Note that the reconstruction constant $C(\Omega, \mathrm{T})$ for (3.1) is given by

$$C(\Omega, \mathrm{T}) = \left(\inf_{\substack{g \in \mathrm{T} \\ \|g\|=1}} \langle \mathcal{S}g, g \rangle \right)^{-1/2}.$$

In particular, $C(\Omega, \mathrm{T}) \to 1$ as $N \to \infty$ for fixed, finite-dimensional T.

Remark 3.1 In typical applications, $\mathrm{T} = \mathrm{span}\{\phi_n\}_{n=1}^{M}$ is the space spanned by the first M elements of a Riesz basis or frame $\{\phi_n\}_{n \in \mathbb{N}}$ for H. Since the number of samples N is usually fixed in applications, it may be more natural to think of decreasing M rather than increasing N. Mathematically, both are equivalent. Whilst noting this, we shall continue to consider increasing N (or later, the bandwidth K), due to its connection with strong convergence.

3.2 Extension to nonuniform samples

Suppose that $\Omega = \{\omega_1, \ldots, \omega_N\} \subseteq \mathbb{R}$ is a sampling scheme based on ordered nonuniform samples $\omega_1 < \ldots < \omega_N$. We now wish to extend the GS framework to this setting, leading to so-called nonuniform generalized sampling (NUGS). Our first step is to replace strong convergence of the partial Fourier series (3.3) by a weaker condition:

Definition 3.2. *Let Ω be a sampling scheme, $\mathcal{S} : \mathrm{H} \to \mathrm{H}$ a bounded linear operator and let* T *be a finite-dimensional subspace of* H. *Suppose that $\mathcal{S}$ satisfies*

(i) for each $f \in \mathrm{H}$, $\mathcal{S}f$ depends only on the sampling data $\{\hat{f}(\omega) : \omega \in \Omega\}$,

(ii) $\mathcal{S}$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle$ and satisfies

$$|\langle \mathcal{S}f, g \rangle| \leq \sqrt{\langle \mathcal{S}f, f \rangle \langle \mathcal{S}g, g \rangle}, \quad \forall f, g \in \mathrm{H}, \tag{3.4}$$

(iii) there exists a positive constant $C_1 = C_1(\Omega, \mathrm{T})$ such that

$$\langle \mathcal{S}f, f \rangle \geq C_1 \|f\|^2, \quad \forall f \in \mathrm{T}, \tag{3.5}$$

Then $\mathcal{S}$ is said to be an admissible sampling operator for the pair (Ω, T).

For convenience, throughout the remainder of the paper we shall assume that C_1 is the largest constant for which (3.5) holds. Given such an operator $\mathcal{S}$, we now also define the constants $C_2 = C_2(\Omega)$ and $C_3 = C_3(\Omega, \mathrm{T})$ by

$$\langle \mathcal{S}f, f \rangle \leq C_2 \|f\|^2, \quad \forall f \in \mathrm{H}. \tag{3.6}$$

and

$$\langle \mathcal{S}f, f \rangle \leq C_3 \|f\|^2, \quad \forall f \in \mathrm{T}, \tag{3.7}$$

Likewise, we assume these constants are the smallest possible. Note that C_2 and C_3 exist since $\mathcal{S}$ is bounded, and we also trivially have that $C_3 \leq C_2$. We remark in passing that we usually want C_2 to be independent of the number of samples N (or more precisely, the bandwidth K of Ω – see Definition 4.1), since, as we see later, it will appear in the error and stability estimates.

The inequalities (3.5) and (3.7) relax the condition of strong convergence of the operator $\mathcal{S}$. Specifically, they ensure that the bilinear form $\langle \mathcal{S}\cdot, \cdot \rangle$ gives rise to an equivalent inner product on T. We remark in passing that in the case of uniform sampling, the operator $\mathcal{S}$ defined by (3.3) is automatically an admissible sampling operator whenever N is sufficiently large, with constants $C_1 \approx 1$ for large N and $C_2 = 1$ for all N.

In the nonuniform setting, there are many potential ways in which one could construct the operator $\mathcal{S}$. In this paper, we shall focus primarily on the following simple construction:

$$\mathcal{S}f(x) = \sum_{n=1}^{N} \mu_n \hat{f}(\omega_n) \mathrm{e}^{2\pi i n x}, \quad x \in [0, 1], \tag{3.8}$$

where $\mu_n > 0$ are particular weights. Observe that $\mathcal{S}$, when defined in this way, automatically satisfies properties (i) and (ii) for an admissible sampling operator. Clearly, in the case uniform sampling, (3.8) reduces to (3.3) when the weights $\mu_n = 1/\epsilon$.

Given a sampling scheme Ω, a finite-dimensional subspace T and an admissible sampling operator $\mathcal{S}$ we now define the NUGS reconstruction by

$$\tilde{f} \in \mathrm{T}, \qquad \langle \mathcal{S}\tilde{f}, g \rangle = \langle \mathcal{S}f, g \rangle, \quad \forall g \in \mathrm{T}, \tag{3.9}$$

and write $F = F_{\Omega,\mathrm{T}}$ for the mapping $f \mapsto \tilde{f}$. As in the uniform case, if $\mathcal{S}$ is given by (3.8), then this is equivalent to the weighted least-squares data fit:

$$\tilde{f} = \underset{g \in \mathrm{T}}{\operatorname{argmin}} \sum_{n=1}^{N} \mu_n \left| \hat{f}(\omega_n) - \hat{g}(\omega_n) \right|^2. \tag{3.10}$$

As we shall see next, the constants C_1, C_2 and C_3 arising from an admissible sampling operator $\mathcal{S}$ determine the stability and quasi-optimality of the resulting NUGS reconstruction via the reconstruction constant $C(\Omega, \mathrm{T})$. We now formally define this constant:

Definition 3.3. *Let $\mathcal{S}$ be an admissible sampling operator with constants C_1 and C_2 given by (3.5) and (3.6) respectively. The ratio $C(\Omega, \mathrm{T}) = \sqrt{C_2/C_1}$ is referred to as the NUGS reconstruction constant.*

3.3 Analysis

We now show existence, uniqueness, stability and quasi-optimality of NUGS:

Theorem 3.4. *Let Ω be a sampling scheme and T a finite-dimensional subspace, and suppose that $\mathcal{S}$ is an admissible sampling operator. Then the reconstruction $F(f) = \tilde{f}$ defined by (3.9) exists uniquely for any $f \in \mathrm{H}$ and we have the sharp bound*

$$\|f - F(f+h)\| \leq \tilde{C}\left(\|f - \mathcal{P}_{\mathrm{T}} f\| + \|h\|\right), \quad \forall f, h \in \mathrm{H}, \tag{3.11}$$

where the constant $\tilde{C}$ is given by

$$\tilde{C} = \tilde{C}(\Omega, \mathrm{T}) = \sup_{\substack{g \in \mathrm{T} \\ g \neq 0}} \left\{ \frac{\|g\|}{\|\mathcal{P}_{\mathcal{S}(\mathrm{T})} g\|} \right\}.$$

Moreover, the constant $\tilde{C}$ satisfies $\tilde{C} \leq C$, where $C = C(\Omega, \mathrm{T})$ is the corresponding reconstruction constant (Definition 3.3).

Note that this theorem extends a previous result [4] to the case of nonuniform samples.

Proof. We first show that $\tilde{C} \leq C$, and in particular, that $\tilde{C} < \infty$. By definition

$$1/\tilde{C} = \inf_{\substack{g \in \mathrm{T} \\ g \neq 0}} \frac{\|\mathcal{P}_{\mathcal{S}(\mathrm{T})} g\|}{\|g\|} = \inf_{\substack{g \in \mathrm{T} \\ g \neq 0}} \sup_{\substack{g' \in \mathrm{T} \\ \mathcal{S}g' \neq 0}} \frac{|\langle g, \mathcal{S} g' \rangle|}{\|g\| \|\mathcal{S} g'\|}.$$

Let $g \in \mathrm{T} \backslash \{0\}$. If $\mathcal{S} g = 0$, then $\langle \mathcal{S} g, g \rangle = 0$ which contradicts the admissibility of $\mathcal{S}$. Hence $\mathcal{S} g \neq 0$. Therefore, we may set $g' = g$ above to get

$$1/\tilde{C} \geq \inf_{\substack{g \in \mathrm{T} \\ g \neq 0}} \frac{\langle \mathcal{S} g, g \rangle}{\|g\| \|\mathcal{S} g\|}.$$

Observe that

$$\|\mathcal{S} g\| = \sup_{\substack{h \in \mathrm{H} \\ \|h\| = 1}} \langle \mathcal{S} g, h \rangle \leq \sqrt{C_2} \sqrt{\langle \mathcal{S} g, g \rangle},$$

where the inequality follows from (3.4) and (3.6). This now gives

$$1/\tilde{C} \geq \frac{1}{\sqrt{C_2}} \inf_{\substack{g \in \mathrm{T} \\ g \neq 0}} \frac{\sqrt{\langle \mathcal{S} g, g \rangle}}{\|g\|},$$

which, upon application of (3.5), yields $\tilde{C} \leq \sqrt{C_2 / C_1} = C$ as required.

To prove the remainder of the theorem, we shall used the techniques of [4] based on the geometric notions of subspace angles and oblique projections. Let $\mathrm{U} = \mathrm{T}$ and $\mathrm{V} = (\mathcal{S}(\mathrm{T}))^{\perp}$. Note that $1/\tilde{C} = \cos(\theta_{\mathrm{UV}^{\perp}})$ is cosine of the subspace angle between U and $\mathrm{V}^{\perp}$ defined by

$$\cos(\theta_{\mathrm{UV}^{\perp}}) = \inf_{\substack{u \in \mathrm{U} \\ \|u\| = 1}} \|\mathcal{P}_{V^{\perp}} u\|.$$

Since $\tilde{C} < \infty$, the subspaces U and V satisfy the so-called subspace condition $\cos(\theta_{\mathrm{UV}^{\perp}}) > 0$. Thus [4, Cor. 3.5] gives

$$\|\mathcal{W}_{\mathrm{UV}} f\| \leq \tilde{C} \|f\|, \quad \forall f \in \mathrm{H}_0,$$

and

$$\|f - \mathcal{W}_{\mathrm{UV}} f\| \leq \tilde{C} \|f - \mathcal{P}_{\mathrm{U}} f\|, \quad \forall f \in \mathrm{H}_0,$$

where $\mathrm{H}_0 = \mathrm{U} \oplus \mathrm{V}$ and $\mathcal{W}_{\mathrm{UV}} : \mathrm{H}_0 \to \mathrm{U}$ is the projection with range U and kernel V.

Hence, to establish (3.11) it remains to show the following: (i) $\mathrm{H}_0 = \mathrm{H}$ and (ii) $\tilde{f} = \mathcal{W}_{\mathrm{UV}} f$, $\forall f \in \mathrm{H}$. For (i), we note that $\mathrm{H}_0 = \mathrm{H}$ provided $\dim(\mathcal{S}(\mathrm{T})) = \dim(\mathrm{T})$ [4, Lem. 3.10]. However, if not then there exists a nonzero $g \in \mathrm{T}$ such that $\mathcal{S}(g) = 0$. As previously observed, this implies that $g = 0$; a contradiction.

For (ii), we first note that

$$\langle \mathcal{W}_{\mathrm{UV}} f, \mathcal{S} g \rangle = \langle f, \mathcal{S} g \rangle, \quad \forall g \in \mathrm{T}.$$

Since $\mathcal{S}$ is self-adjoint, it follows that $\mathcal{W}_{\mathrm{UV}} f$ satisfies the same conditions (3.9) as $\tilde{f}$. Thus, it remains only to show that $\tilde{f}$ is unique. However, if not then we find that there is a nonzero $g \in \mathrm{T} \cap \mathcal{S}(\mathrm{T})^{\perp} = \mathrm{U} \cap \mathrm{V}$. But then $\cos(\theta_{\mathrm{UV}^{\perp}}) = 0$, and this contradicts the fact that U and V obey the subspace condition. $\square$

Theorem 3.4 confirms that admissibility of $\mathcal{S}$ is sufficient for quasi-optimality and stability of the reconstruction $\tilde{f}$ up to the magnitude of the reconstruction constant C. Note that the result is true under the slightly weaker assumption $\tilde{C} < \infty$ (which is of course implied by $C_1 > 0$ and $C_2 < \infty$). However, the constant $\tilde{C}$ is rather difficult to work with in practice. First, it is hard to estimate $\tilde{C}$ for typical reconstruction spaces. Second, it does not give a useful bound for the condition number of the matrix of the linear system (3.9) which, as we discuss next, is important from a computational perspective.

Remark 3.5 Although we shall assume throughout the remainder of the paper that $\mathcal{S}$ takes the form (3.8), the results of this section do not require this. They only assume that $\mathcal{S}$ is admissible in the sense of Definition 3.2, and we note here that the first condition (i), i.e. that the samples are one-dimensional Fourier samples in the Hilbert space $\mathrm{H} = \mathrm{L}^2(0,1)$, is merely for convenience. Crucially for future work, this means that the results in this section easily generalise to multidimensional Fourier measurements, as well as to other sampling problems. Moreover, even in one dimension it allows for the possibility of more general (e.g. nondiagonal) forms $\mathcal{S}$ than (3.8).

3.4 Computation of the reconstruction and the reconstruction constant

We now discuss implementation of NUGS. Recall that if $\mathcal{S}$ is given by (3.8), then (3.9) is equivalent to (3.10). In particular, if $\{\phi_m\}_{m=1}^{P}$ is a basis for T, and if the reconstruction $\tilde{f}$ is given by

$$\tilde{f} = \sum_{m=1}^{P} a_m \phi_m,$$

then the vector of coefficients $a = (a_1, \ldots, a_P)^{\top}$ is the least squares solution of the $N \times P$ linear system

$$Aa \approx b, \tag{3.12}$$

where $b = (b_1, \ldots, b_N)^{\top}$ and $A \in \mathbb{C}^{N \times P}$ have entries

$$b_n = \sqrt{\mu_n} \hat{f}(\omega_n), \quad A_{n,m} = \sqrt{\mu_n} \widehat{\phi_m}(\omega_n), \quad n = 1, \ldots, N, \; m = 1, \ldots, P. \tag{3.13}$$

Thus, once a basis for T is specified, $\tilde{f}$ can be computed by solving the least squares problem (3.12). The computational cost in doing so is proportional to the condition number of A, which determines the number of iterations required in an iterative solver such as conjugate gradients, multiplied by the cost of performing matrix-vector operations with A and its adjoint A^*. For the former, we have the following:

Theorem 3.6. *Let $\{g_m\}_{m=1}^P$ be a basis for T and suppose that A is defined by (3.13). Then the constants $C_3(\Omega,\mathrm{T})$ and $C_1(\Omega,\mathrm{T})$ are the maximal and minimal eigenvalues of the matrix pencil $\{A^*A, B\}$, where $B \in \mathbb{C}^{P\times P}$ is the Gram matrix for $\{\phi_m\}_{m=1}^P$. In particular, the condition number of A satisfies*

$$C_w(\Omega,\mathrm{T})/\sqrt{\kappa(B)} \leq \kappa(A) \leq \sqrt{\kappa(B)}C_w(\Omega,\mathrm{T}),$$

where $C_w(\Omega,\mathrm{T}) = \sqrt{C_3(\Omega,\mathrm{T})/C_1(\Omega,\mathrm{T})}$. Moreover, if $\{g_m\}_{m=1}^P$ is an orthonormal basis then

$$C_3(\Omega,\mathrm{T}) = \sigma^2_{\max}(A), \quad C_1(\Omega,\mathrm{T}) = \sigma^2_{\min}(A),$$

and $\kappa(A) = C_w(\Omega,\mathrm{T})$.

Proof. The proof is similar to that given in [4] and hence is omitted. ☐

This theorem asserts that, provided a Riesz or orthonormal basis is chosen for T (so that $\kappa(B)$ is small), the condition number of A is small precisely when $C_w(\Omega,\mathrm{T})$ is also small. In this case, the reconstruction $\tilde{f}$ can be computed using a correspondingly small number of iterations.

Note that this theorem also asserts that $C_w(\Omega,\mathrm{T})$ can be computed. Unfortunately, $C_w(\Omega,\mathrm{T})$ provides only a lower bound for the reconstruction constant $C(\Omega,\mathrm{T})$, and thus computing $C_w(\Omega,\mathrm{T})$ does not give rise to an estimate for the constant in the error bound (3.11). Nevertheless, the fact that $C_w(\Omega,\mathrm{T})$ is computable means that $C(\Omega,\mathrm{T})$ can in fact be numerically approximated via the following limiting process:

Lemma 3.7. *Suppose that Ω is finite and let $\mathcal{S} : \mathrm{H} \to \mathrm{H}$ be a linear operator satisfying conditions (i) and (ii) of Definition 3.2. Let T_N, $N \in \mathbb{N}$, be a sequence of finite-dimensional reconstruction spaces such that the corresponding orthogonal projections $\mathcal{P}_N = \mathcal{P}_{\mathrm{T}_N}$ converge strongly to the identity on H. Then*

$$C_2(\Omega) = \lim_{N\to\infty} C_3(\Omega,\mathrm{T}_N).$$

In particular, $C_2(\Omega)$ can be approximated to arbitrary accuracy by taking N sufficiently large.

Proof. Note first that $C_3(\Omega,\mathrm{T}_N) \leq C_2(\Omega)$. Let $f \in \mathrm{H}$, $\|f\| = 1$. Then

$$\begin{aligned}\langle \mathcal{S}f, f\rangle &= \langle \mathcal{S}\mathcal{P}_N f, \mathcal{P}_N f\rangle + \langle \mathcal{S}(f - \mathcal{P}_N f), \mathcal{P}_N f\rangle + \langle \mathcal{S}f, f - \mathcal{P}_N f\rangle \\ &\leq C_3(\Omega,\mathrm{T}_N) + 2\sqrt{C_2(\Omega)}\sqrt{\langle \mathcal{S}(f - \mathcal{P}_N f), f - \mathcal{P}_N f\rangle}.\end{aligned}$$

Thus,

$$C_3(\Omega,\mathrm{T}_N) \leq C_2(\Omega) \leq C_3(\Omega,\mathrm{T}_N) + 2\sqrt{C_2(\Omega)} \sup_{\substack{f\in\mathrm{H} \\ \|f\|=1}} \sqrt{\langle \mathcal{S}(f - \mathcal{P}_N f), f - \mathcal{P}_N f\rangle}.$$

It suffices to prove that the final term tends to zero as $N \to \infty$.

The operator $\mathcal{S}$ is linear and, for any g, $\mathcal{S}g$ depends only on the finite set of values $\hat{g}(\omega)$, $\omega \in \Omega$. Therefore, $\mathcal{S}$ is bounded and has finite rank. The result now follows immediately from this and the strong convergence $\mathcal{P}_N \to \mathcal{I}$. ☐

Since $C_2(\Omega)$ can always be approximated for finite Ω, one can always numerically estimate the reconstruction constant $C(\Omega, \mathrm{T})$ and therefore guarantee stability and quasi-optimality of the reconstruction *a priori*. Note that different choices of the sequence T_N, $N \in \mathbb{N}$, in the limiting process may give faster convergence to this limit. We shall not discuss this issue. We also note that this limiting process may be avoided altogether in the case where $\mathcal{S}$ is given by (3.8) with appropriate weights, and where the samples satisfy an appropriate density condition (see Definition 4.1 below) or arise from a Fourier frame. This is discussed in §4.3.

Remark 3.8 As mentioned, efficient computation of $\tilde{f}$ relies on a fast algorithm for performing matrix-vector computations with A and A^*. The existence of such algorithms depends critically on the choice of the reconstruction space T. Fortunately, in the important case of wavelets, fast algorithms can be incorporated. These are based on Nonuniform Fast Fourier Transforms (NUFFTs) and fast wavelet transforms. Since the focus of this paper is primarily on theory, however, we shall not discuss this in further detail. A future work describing the multidimensional case will address this topic.

4 A generalized sampling theorem for nonuniform samples

Having introduced NUGS, we now establish one of the main results of the paper. Namely, we prove a generalized sampling theorem which asserts that stable, quasi-optimal reconstruction is possible for any fixed T under appropriate conditions on the nonuniform sampling scheme Ω.

We shall consider two sampling scenarios. First, sampling schemes Ω subject to appropriate density and bandwidth conditions. Second, sampling schemes arising from Fourier frames. These scenarios are the topics of the next two subsections.

4.1 (K, δ)-dense sampling schemes

Definition 4.1. *Let $K > 0$ and $0 < \delta < 1$ and $\omega_1 < \omega_2 < \ldots < \omega_N$. The sampling scheme $\Omega = \{\omega_n : n = 1, \ldots, N\}$ has bandwidth K and density δ if $\Omega \subseteq [-K, K]$ and*

$$\max_{n=0,\ldots,N} \{\omega_{n+1} - \omega_n\} \leq \delta,$$

where $\omega_0 = \omega_N - 2K$ and $\omega_{N+1} = \omega_1 + 2K$. In this case, we say that Ω is (K, δ)-dense.

Our main result in this section is to show that, for an arbitrary fixed reconstruction space T, (K, δ)-density for suitably large K and small δ ensures stable reconstruction. This holds provided the weights μ_n in (3.8) are chosen according to the following strategy:

$$\mu_n = \frac{1}{2}(\omega_{n+1} - \omega_{n-1}), \quad n = 1, \ldots, N, \tag{4.1}$$

where, as above, we set $\omega_0 = \omega_N - 2K$ and $\omega_{N+1} = \omega_1 + 2K$ (other choices of weights are possible, but we shall not address this issue). Note that our (K, δ)-density condition is similar to the condition used by Feichtinger, Gröchenig & Strohmer in nonuniform sampling [21], as well as that found in Potts & Tasche on the numerical stability on NUFFTs [33].

We commence with the following lemma:

Lemma 4.2. *Let $\Omega = \{\omega_1, \ldots, \omega_N\}$ be (K, δ)-dense and suppose that $\mu_1, \ldots, \mu_N$ are given by (4.1). Then for any nonzero $f \in \mathrm{L}^2(0, 1)$ we have*

$$\left(\sqrt{1 - \|\hat{f}\|^2_{\mathbb{R}\setminus I}/\|f\|^2} - \delta\right)^2 \|f\|^2 \leq \sum_{n=1}^{N} \mu_n |\hat{f}(\omega_n)|^2 \leq (1+\delta)^2 \|f\|^2,$$

where $I = (-K + \frac{1}{2}\delta, K - \frac{1}{2}\delta)$*, and* $\|\hat{f}\|^2_{\mathbb{R}\setminus I} = \int_{\mathbb{R}\setminus I} |\hat{f}(\omega)|^2 \,\mathrm{d}\omega$.

Note that this lemma is an extension of a result of Gröchenig [22] to the case where the number of samples N is finite. Gröchenig's result is obtained in the limit $N, K \to \infty$. We remark also that the lower bound is strictly less than $(1-\delta)^2$ for any nonzero f. This follows from the observation that since f is supported in $[0,1]$, $\hat{f}$ cannot have compact support. However, the lower bound converges to $(1-\delta)^2$ as the bandwidth K is increased. In other words, N Fourier samples with density $\delta < 1$ and appropriately large bandwidth K are sufficient to control $\|f\|$. As we shall see below, this observation leads to the main result in this section.

Proof of Lemma 4.2. First, let us define a function $F \in \mathrm{L}^2\left(-\frac{1}{2}, \frac{1}{2}\right)$ such that $F(x) = f(x + 1/2)$. Since $|\hat{F}(\omega)| = |\hat{f}(\omega)|$, and also $\|F\| = \|f\|$, it is enough to prove the theorem for the function F. We now proceed similarly to as in [22]. Let $z_n = \frac{1}{2}(\omega_{n-1} + \omega_n)$ and write

$$\chi(\omega) = \sum_{n=1}^{N} \hat{F}(\omega_n) \mathbb{I}_{[z_n, z_{n+1})}(\omega),$$

so that

$$S^2 = \sum_{n=1}^{N} \mu_n |\hat{F}(\omega_n)|^2 = \int_{z_1}^{z_{N+1}} |\chi(x)|^2 \,\mathrm{d}x = \|\chi\|^2_J,$$

where $J = (z_1, z_{N+1})$ and $\|\cdot\|_J$ denotes the L^2-norm over J. Hence

$$\|\hat{F}\|_J - \|\hat{F} - \chi\|_J \le S \le \|\hat{F}\|_{\mathbb{R}} + \|\hat{F} - \chi\|_J. \tag{4.2}$$

Using Wirtinger's inequality [22, Lem. 1], we find that

$$\begin{aligned}
\|\hat{F} - \chi\|^2_J &= \sum_{n=1}^{N} \int_{z_n}^{z_{n+1}} \left|\hat{F}(\omega) - \hat{F}(\omega_n)\right|^2 \,\mathrm{d}\omega \\
&= \sum_{n=1}^{N} \left(\int_{z_n}^{\omega_n} + \int_{\omega_n}^{z_{n+1}}\right) \left|\hat{F}(\omega) - \hat{F}(\omega_n)\right|^2 \,\mathrm{d}\omega \\
&\le \sum_{n=1}^{N} \left(\frac{4(\omega_n - z_n)^2}{\pi^2} \int_{z_n}^{\omega_n} + \frac{4(z_{n+1} - \omega_n)^2}{\pi^2} \int_{\omega_n}^{z_{n+1}}\right) \left|\frac{\mathrm{d}}{\mathrm{d}\omega}\hat{F}(\omega)\right|^2 \,\mathrm{d}\omega \\
&\le \frac{\delta^2}{\pi^2} \int_J \left|\frac{\mathrm{d}}{\mathrm{d}\omega}\hat{F}(\omega)\right|^2 \,\mathrm{d}\omega,
\end{aligned}$$

where the final inequality follows from the (K, δ)-density of the samples. Since differentiation in Fourier space corresponds to multiplication by $(-2\pi \mathrm{i} x)$ in physical space, we conclude that

$$\|\hat{F} - \chi\|_J \le 2\delta \left\|\widehat{F_1}\right\|_J \le 2\delta \left\|\widehat{F_1}\right\|_{\mathbb{R}}, \qquad F_1(x) = xF(x).$$

The Fourier transform satisfies $\|\hat{g}\|_{\mathbb{R}} = \|g\|_{\mathbb{R}}$, $\forall g \in \mathrm{L}^2(\mathbb{R})$. Since F is supported in $[-1/2, 1/2]$, we deduce that

$$\|\hat{F} - \chi\|_J \le 2\delta \|F_1\| \le \delta \|F\|. \tag{4.3}$$

Substituting this into the right-hand side of (4.2), we get $S \le (1+\delta)\|F\|$, which gives the upper bound. For the lower bound, we first note that $I \subseteq J$. Hence, using (4.2) and (4.3) we get

$$S \ge \|\hat{F}\|_I - \delta\|F\| \ge \sqrt{\|\hat{F}\|^2 - \|\hat{F}\|^2_{\mathbb{R}\setminus I}} - \delta\|F\|,$$

and the lower bound follows. $\square$

Definition 4.3. *Let* $\mathrm{T} \subseteq \mathrm{H}$. *The* z*-residual of* T *is the quantity*

$$E(\mathrm{T}, z) = \sup\left\{ \|\hat{f}\|_{\mathbb{R}\setminus(-z,z)} : f \in \mathrm{T},\ \|f\| = 1 \right\}, \quad z \in [0, \infty). \tag{4.4}$$

Note that $E(\mathrm{T}, z) \leq 1$, $\forall z$ and any T, since $\|\hat{f}\| = \|f\|$.

Lemma 4.4. *Let* $\mathrm{T} \subseteq \mathrm{H}$ *be finite-dimensional. Then* $E(\mathrm{T}, z) \to 0$ *monotonically as* $z \to \infty$.

Proof. Clearly $E(\mathrm{T}, z)$ is monotonically decreasing in z. Moreover, for any fixed $f \in \mathrm{T}$, we have $\|\hat{f}\|_{\mathbb{R}\setminus(-z,z)} \to 0$ as $z \to \infty$. The result now follows immediately since T is finite-dimensional. $\square$

The relevance of the z-residual is that it gives an upper bound for the reconstruction constants $C(\Omega, \mathrm{T})$ for an arbitrary subspace T. Combining the previous two lemmas, we immediately obtain our main result of this section:

Theorem 4.5. *Let* $\mathrm{T} \subseteq \mathrm{H}$ *be finite-dimensional and let* Ω *be* (K, δ)*-dense, where*

$$\delta < \sqrt{1 - E(\mathrm{T}, K - \tfrac{1}{2})^2}.$$

Let $\mathcal{S}$ *be given by (3.8) with weights (4.1). Then* $\mathcal{S}$ *is admissible with reconstruction constant* $C(\Omega, \mathrm{T})$ *(see Definition 3.3) satisfying*

$$C(\Omega, \mathrm{T}) \leq \frac{1 + \delta}{\sqrt{1 - E(\mathrm{T}, K - \frac{1}{2})^2} - \delta}. \tag{4.5}$$

Proof. The upper bound in Lemma 4.2 immediately gives $C_2(\Omega) \leq (1 + \delta)^2$. For $C_1(\Omega, \mathrm{T})$ we set $f = g \in \mathrm{T}$ in Lemma 4.2, and then apply the definition of $E(\mathrm{T}, z)$ to get

$$C_1(\Omega, \mathrm{T}) \geq \left(\sqrt{1 - E(\mathrm{T}, K - \tfrac{1}{2}\delta)^2} - \delta \right)^2.$$

The result now follows from monotonicity of $E(\mathrm{T}, z)$ and the definition of $C(\Omega, \mathrm{T})$. $\square$

The main consequence of this theorem is as follows. For a fixed reconstruction space T, the reconstruction constant $C(\Omega, \mathrm{T})$ can be made arbitrarily close to $\frac{1+\delta}{1-\delta}$ by taking K sufficiently large. Thus, even with highly nonuniform samples, we are guaranteed a stable reconstruction for large enough bandwidth K provided the density condition $\delta < 1$ holds, with the precise level of stability controlled primarily by how close δ is to one.

Another important aspect of the Theorem 4.5 is the nature of the bound (4.5). The right-hand side separates geometric properties of the sampling scheme Ω, i.e. the density δ, from intrinsic properties of the reconstruction space T, i.e. the z-residual $E(\mathrm{T}, z)$. Hence, by analysing the z-residual for each particular choice of T, we can guarantee stable, quasi-optimal reconstruction for *all* sampling schemes Ω with $\delta < 1$ and appropriate bandwidth K. This is how we shall proceed in §5 when we consider wavelet reconstruction spaces.

4.2 Sampling at the critical density: the frame case

As commented previously, it is commonplace in nonuniform sampling to assume that the sampling points $\omega_{-N}, \omega_{1-N}, \ldots, \omega_N$, which we now index from $-N$ to N, arise from an infinite sequence $\{\omega_n\}_{n\in\mathbb{Z}}$ which gives rise to a Fourier frame $\{\mathrm{e}^{2\pi \mathrm{i}\omega_n \cdot}\mathbb{I}_{[0,1]}(\cdot)\}_{n\in\mathbb{Z}}$ for H. The theory presented in the previous section, which establishes stable reconstruction for arbitrary (K,δ)-dense sampling schemes Ω, does not require this. As discussed in §1, this approach has a number of advantages over the frame approach.

Unfortunately, all the bounds for $C(\Omega, \mathrm{T})$ found in the previous section decline as $\delta \to 1^-$, and are infinitely large at the critical value $\delta = 1$. This result is sharp in the sense that there are countable nonuniform sampling schemes $\Omega = \{\omega_n\}_{n\in\mathbb{Z}}$ with density $\delta = 1$ which are not complete (for an example, see [14] or [42]), and for which one therefore cannot expect stable or quasi-optimal reconstructions. However, it is clear from considering uniform samples $\Omega = \{n\}_{n\in\mathbb{Z}}$ that density $\delta = 1$ is permissible in some cases. The standard approach to handle this "critical" density is to assume that the samples $\Omega = \{\omega_n\}_{n\in\mathbb{Z}}$ give rise to a Fourier frame. As we show next, stable reconstruction is also possible in this setting within the NUGS framework.

4.2.1 Background and notation

Let $\{\omega_n\}_{n\in\mathbb{Z}}$ give rise to a Fourier frame for H. In other words, there exist constants $0 < A \leq B < \infty$ (the frame constants) such that

$$A\|f\|^2 \leq \sum_{n\in\mathbb{Z}} |\hat{f}(\omega_n)|^2 \leq B\|f\|^2, \quad \forall f \in \mathrm{H}.$$

Note that the operator

$$\mathcal{S} : \mathrm{H} \to \mathrm{H}, f \mapsto \sum_{n\in\mathbb{Z}} \hat{f}(\omega_n) \mathrm{e}^{2\pi \mathrm{i}\omega_n \cdot}\mathbb{I}_{[0,1]}(\cdot), \tag{4.6}$$

the so-called frame operator, is well-defined, linear, bounded and invertible, and satisfies

$$A\|f\|^2 \leq \langle \mathcal{S}f, f\rangle \leq B\|f\|^2, \quad \forall f \in \mathrm{H}.$$

Moreover, the truncated operators $\mathcal{S}_N : f \mapsto \sum_{n=-N}^{N} \hat{f}(\omega_n)\mathrm{e}^{2\pi \mathrm{i}\omega_n \cdot}\mathbb{I}_{[0,1]}(\cdot)$ converge strongly to $\mathcal{S}$ on H as $N \to \infty$.

It shall be important later to have conditions under which a sequence $\{\omega_n\}_{n\in\mathbb{Z}}$ gives rise to a Fourier frame. Fortunately, in the one-dimensional setting of this paper, a near-characterization is known. To state this, we first require several definitions:

(i) A sequence of points $\lambda_k \in \mathbb{R}$, $k \in I$, is called *separated* if

$$|\lambda_k - \lambda_j| \geq \eta, \quad j \neq k.$$

for some $\eta > 0$. If $\{\lambda_k\}_{k\in I}$ is a finite union of separated sets, i.e.

$$\{\lambda_k\}_{k\in I} = \bigcup_{n=1}^{K} \{\lambda_k\}_{k\in I_n},$$

where each $\{\lambda_k\}_{k\in I_n}$ is separated, then we call $\{\lambda_k\}_{k\in I}$ *relatively* separated.

(ii) For a sequence $\{\omega_n\}_{n\in\mathbb{Z}}$, the lower Beurling density is defined by

$$D^- = \lim_{r\to\infty} \frac{n^-(r)}{r}, \qquad n^-(r) = \min_{t\in\mathbb{R}} |\{n \in \mathbb{Z} : \omega_n \in (t, t+r)\}| .$$

The following theorem, due to Jaffard [27] and Seip [37], gives an almost characterization of Fourier frames in terms of relative separation and the Beurling density:

Theorem 4.6. *If $\{\omega_n\}_{n\in\mathbb{N}}$ is relatively separated and $D^- > 1$ then $\{\mathrm{e}^{2\pi\mathrm{i}\omega_n\cdot}\mathbb{I}_{[0,1]}(\cdot)\}_{n\in\mathbb{Z}}$ forms a frame for* H*. Conversely, If $\{\mathrm{e}^{2\pi\mathrm{i}\omega_n\cdot}\mathbb{I}_{[0,1]}(\cdot)\}_{n\in\mathbb{Z}}$ forms a frame for* H *then $D^- \geq 1$ and $\{\omega_n\}_{n\in\mathbb{Z}}$ is relatively separated.*

Note that there exist both relatively separated sequences with $D^- = 1$ which form frames and relatively separated sequences with $D^- = 1$ which do not. See [14] for details.

4.2.2 Stable reconstructions from frame samples

Let $\Omega = \Omega_N = \{\omega_n : |n| \leq N\}$ be the first $2N+1$ entries of a sequence $\{\omega_n : n \in \mathbb{Z}\}$ that forms a Fourier frame. According to the results of §3.2 and §3.3, stable reconstruction is possible provided an admissible sampling operator exists. Fortunately, this is always the case:

Theorem 4.7. *Let* T *be a finite-dimensional subspace of* H*, and suppose that $\Omega_N = \{\omega_n : |n| \leq N\}$, where $\{\omega_n : n \in \mathbb{Z}\}$ gives rise to a Fourier frame. Then the partial frame operator*

$$\mathcal{S}_N : f \mapsto \sum_{n=-N}^{N} \hat{f}(\omega_n)\mathrm{e}^{2\pi\mathrm{i}\omega_n\cdot}, \tag{4.7}$$

is admissible for all sufficiently large N. Specifically,

$$C(\Omega, \mathrm{T}) \leq \frac{\sqrt{B}}{\sqrt{A - \tilde{E}(\mathrm{T}, N)^2}}, \tag{4.8}$$

where A and B are the frame constants and

$$\tilde{E}(\mathrm{T}, N)^2 = \sup\left\{ \sum_{|n|>N} |\hat{f}(\omega_n)|^2 : \; f \in \mathrm{T}, \|f\| = 1 \right\}. \tag{4.9}$$

Proof. The operator $\mathcal{S}_N$ trivially satisfies conditions (i) and (ii) of Definition 3.2. For the upper bound (3.6) we merely note that $\langle \mathcal{S}_N f, f\rangle \leq \langle \mathcal{S} f, f\rangle \leq B\|f\|^2$, where $\mathcal{S}$ is the frame operator (4.6). Moreover, since $\mathcal{S}_N \to \mathcal{S}$ strongly and T is finite-dimensional, (3.5) holds (with appropriate C_1) for all large N. Specifically, for $f \in \mathrm{T}$ we have

$$\langle \mathcal{S}_N f, f\rangle = \langle \mathcal{S} f, f\rangle - \langle (\mathcal{S} - \mathcal{S}_N) f, f\rangle \geq A\|f\|^2 - \sum_{|n|>N} |\hat{f}(\omega_n)|^2 \geq \left(A - \tilde{E}(\mathrm{T}, N)^2\right) \|f\|^2,$$

which gives $C_1(\Omega, \mathrm{T}) \geq (A - \tilde{E}(\mathrm{T}, N))$. The bound (4.8) now follows from the definition of C. $\square$

Note that GS was extended to frame samples in [4]. When the sampling operator is given by (4.7), NUGS reduces to GS. This section on frame samples is included primarily for completeness. The novel results in the paper concerning frames come in the next two sections when we obtain estimates for the reconstruction constant $C(\Omega, \mathrm{T})$.

4.3 Summary and estimation of constants

Let us now sum up. For reconstructions from nonuniform Fourier samples, we can distinguish two cases. When the samples $\Omega = \{\omega_n : n = 1, \ldots, N\}$ are (K,δ)-dense, the results of §4.1 establish stable reconstruction with simple, numerically-verifiable, bounds for $C(\Omega, \mathrm{T})$. Specifically, we may compute $C_1(\Omega, \mathrm{T})$ via Theorem 3.6, and use the bound $C_2(\Omega) \leq (1+\delta)^2$ obtained in Theorem 4.5 to give the computable estimate

$$C(\Omega, \mathrm{T}) \leq C_B(\Omega, \mathrm{T}) = \frac{1+\delta}{\sqrt{C_1(\Omega, \mathrm{T})}}.$$

Conversely, if the samples are not (K,δ)-dense (e.g. if $\delta = 1$), but arise from a Fourier frame, then as shown in the previous section, stable reconstruction is also possible. Moreover, we have the estimate

$$C(\Omega, \mathrm{T}) \leq C_B(\Omega, \mathrm{T}) = \frac{\sqrt{B}}{\sqrt{C_1(\Omega, \mathrm{T})}},$$

in this setting. Provided the upper frame bound B is known, this estimate can be computed. If B is unknown (which is often the case in practice), then we may use the limiting process described in Lemma 3.7 to compute $C_2(\Omega)$, and therefore $C(\Omega, \mathrm{T})$, to arbitrary accuracy. Note that the same process can also be used in the case of (K,δ)-dense samples. But the improvement in doing so is likely marginal over the estimate $C_2(\Omega) \leq (1+\delta)^2 \leq 4$ (recall that $\delta \leq 1$).

5 Reconstructions in wavelets

In the previous section we established that stable, quasi-optimal reconstruction in arbitrary subspaces T is possible, provided in the (K,δ)-dense case the bandwidth K is taken sufficiently large or in the frame case the parameter N is chosen appropriately large. We now turn our attention to the question of precisely how large K and N need to be for the important case where T consists the first M terms of a wavelet basis. Our main result is to show that K (or N) needs to scale linearly in M to ensure stable, quasi-optimal reconstruction in this setting.

5.1 Preliminaries

Our interest lies in wavelet bases on the interval $[0,1]$. Following [30], we consider three standard constructions – periodic, folded and boundary wavelets – which will be introduced in the next three subsections. First, however, we recall the definition of a multiresolution analysis (MRA).

Definition 5.1. *A multiresolution analysis of* $\mathrm{L}^2(\mathbb{R})$ *generated by a scaling function* $\phi \in \mathrm{L}^2(\mathbb{R})$ *is a nested sequence of closed subspaces* $\{0\} \subseteq \cdots \subseteq V_{-2} \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq \mathrm{L}^2(\mathbb{R})$ *such that*

(i) $\cup_{j\in\mathbb{Z}} V_j = \mathrm{L}^2(\mathbb{R})$ *and* $\cap_{j\in\mathbb{Z}} V_j = \{0\}$,

(ii) for all $j \in \mathbb{Z}$, $f(\cdot) \in V_j$ *if and only if* $f(2\cdot) \in V_{j+1}$,

(iii) the collection $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ *forms a Riesz basis for* V_0.

Recall that a system $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ forms a Riesz basis for V_0 if and only if there exists constants $d_1, d_2 > 0$ such that

$$d_1 \sum_{k\in\mathbb{Z}} |\alpha_k|^2 \leq \left\| \sum_{k\in\mathbb{Z}} \alpha_k \phi(\cdot - k) \right\|^2 \leq d_2 \sum_{k\in\mathbb{Z}} |\alpha_k|^2, \quad \forall \{\alpha_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z}),$$

and $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ forms an orthonormal basis for V_0 if and only if $d_1 = d_2 = 1$. We recall also that this is equivalent to the condition

$$d_1 \leq \sum_{k\in\mathbb{Z}} \left|\hat{\phi}(k+\omega)\right|^2 \leq d_2, \quad a.e.\ \omega \in [0,1]. \tag{5.1}$$

In particular, the optimal Riesz basis constants are given by

$$d_1 = \operatorname*{essinf}_{\omega\in[0,1]} \sum_{k\in\mathbb{Z}} \left|\hat{\phi}(k+\omega)\right|^2, \qquad d_2 = \operatorname*{esssup}_{\omega\in[0,1]} \sum_{k\in\mathbb{Z}} \left|\hat{\phi}(k+\omega)\right|^2.$$

5.1.1 Periodic wavelets

Suppose that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a wavelet basis of $\mathrm{L}^2(\mathbb{R})$ associated to an MRA with scaling function ϕ. Define the periodizing operation

$$f(x) \mapsto f^{\mathrm{per}}(x) = \sum_{k\in\mathbb{Z}} f(x+k), \tag{5.2}$$

and let $\psi^{\mathrm{per}}_{j,k}$ and $\phi^{\mathrm{per}}_{j,k}$ be the corresponding periodic wavelets and scaling functions. Define the periodized MRA spaces

$$V^{\mathrm{per}}_j = \operatorname{span}\left\{\phi^{\mathrm{per}}_{j,k} :\ k = 0,\ldots,2^j - 1\right\}, \quad W^{\mathrm{per}}_j = \operatorname{span}\left\{\psi^{\mathrm{per}}_{j,k} :\ k = 0,\ldots,2^j - 1\right\}.$$

Note that the maximal index k is finite, since $\phi^{\mathrm{per}}_{j,k+2^j} = \phi^{\mathrm{per}}_{j,k}$ and likewise for $\psi^{\mathrm{per}}_{j,k}$.

Now let $J \in \mathbb{N}_0$ be given. Then

$$\mathrm{L}^2(0,1) = \overline{V^{\mathrm{per}}_J \oplus W^{\mathrm{per}}_J \oplus W^{\mathrm{per}}_{J+1} \oplus \cdots},$$

and we may therefore introduce the finite-dimensional reconstruction space T by truncating the right-hand side:

$$\mathrm{T} = V^{\mathrm{per}}_J \oplus W^{\mathrm{per}}_J \oplus W^{\mathrm{per}}_{J+1} \oplus \cdots \oplus W^{\mathrm{per}}_{R-1}. \tag{5.3}$$

Note that $\dim(\mathrm{T}) = 2^R$. Since the original wavelets have an MRA, we also have that

$$\mathrm{T} = V^{\mathrm{per}}_R = \operatorname{span}\left\{\phi^{\mathrm{per}}_{R,k} : k = 0,\ldots,2^R - 1\right\}.$$

Our primary interest in this paper lies with wavelet bases having compact support. Without loss of generality, we now suppose that $\operatorname{supp}(\phi) \subseteq [-p+1, p]$ for $p \in \mathbb{N}$. Note the following: if $\operatorname{supp}(f) \subseteq [0,1]$ then $f(x) = f^{\mathrm{per}}(x)$ for $x \in [0,1]$. In particular, since

$$\operatorname{supp}(\phi_{R,k}) = [(k-p+1)/2^R, (k+p)/2^R],$$

we have that $\phi^{\mathrm{per}}_{R,k}(x) = \phi_{R,k}(x)$, $x \in [0,1]$, whenever $k = p,\ldots,2^R - p - 1$. Hence we may decompose the space T into

$$\mathrm{T} = \mathrm{T}^{\mathrm{left}} \oplus \mathrm{T}^i \oplus \mathrm{T}^{\mathrm{right}}, \tag{5.4}$$

where

$$\mathrm{T}^i = \operatorname{span}\left\{\phi_{R,k} : k = p,\ldots,2^R - p - 1\right\},$$

contains interior scaling functions with support in $(0,1)$ and

$$\mathrm{T}^{\mathrm{left}} = \operatorname{span}\left\{\phi^{\mathrm{per}}_{R,k}\mathbb{I}_{[0,1]} : k = 0,\ldots,p-1\right\}, \quad \mathrm{T}^{\mathrm{right}} = \operatorname{span}\left\{\phi^{\mathrm{per}}_{R,k}\mathbb{I}_{[0,1]} : k = 2^R - p,\ldots,2^R - 1\right\},$$

contains the periodized scaling functions. Here $\mathbb{I}_{[0,1]}$ is the indicator function of the interval $[0,1]$. Whilst not strictly necessary at this point, we add this function to the definitions of $\mathrm{T}^{\mathrm{left}}$ and $\mathrm{T}^{\mathrm{right}}$ so as to clarify that they are to be considered as subspaces of $\mathrm{H} = \{g \in \mathrm{L}^2(\mathbb{R}) : \mathrm{supp}(g) \subseteq [0,1]\}$ in our setting, and not $\mathrm{L}^2(\mathbb{R})$.

Remark 5.2 The stipulation that $\mathrm{supp}(\phi) \subseteq [-p+1, p]$ with $p \in \mathbb{N}$ makes little difference (besides affecting the constant) to the main result we establish in this section regarding $C(\Omega, \mathrm{T})$ with T as above. The key point is that ϕ should have compact support. In which case we can always find $p \in \mathbb{N}$ such that $\mathrm{supp}(\phi) \subseteq [-p+1, p]$.

5.1.2 Folded wavelets

The above process of periodization for creating wavelet bases on $[0,1]$ is widely used in standard implementations, since it is extremely simple. However, vanishing moments of the wavelets are lost due to the enforcement of periodic boundary conditions. This effectively introduces a discontinuity of the signal at the boundaries, and translates into lower approximation orders.

Folded wavelets remove the artificial signal discontinuity introduced by periodization and allow for one vanishing moment to be retained. This is achieved via the folding operation

$$f(x) \mapsto f^{\mathrm{fold}}(x) = \sum_{k\in\mathbb{Z}} f(x-2k) + \sum_{k\in\mathbb{Z}} f(2k-x). \tag{5.5}$$

This approach is most commonly used for the CDF wavelets [15]. In this case, one obtains biorthogonal bases of wavelets for H. Note that we have

$$V_j^{\mathrm{fold}} = \mathrm{span}\left\{\phi_{j,k}^{\mathrm{fold}} :\ k = 0, \dots, 2^j - \iota\right\}, \quad W_j^{\mathrm{fold}} = \mathrm{span}\left\{\psi_{j,k}^{\mathrm{fold}} :\ k = 0, \dots, 2^j - 1\right\},$$

where ι takes value 0 if the wavelets are symmetric about $x = 1/2$ and 1 if they are antisymmetric. Much as before, we define the finite-dimensional reconstruction space

$$\mathrm{T} = V_J^{\mathrm{fold}} \oplus W_J^{\mathrm{fold}} \oplus W_{J+1}^{\mathrm{fold}} \oplus \cdots \oplus W_{R-1}^{\mathrm{fold}}, \tag{5.6}$$

and note that

$$\mathrm{T} = V_R^{\mathrm{fold}} = \mathrm{span}\left\{\phi_{R,k}^{\mathrm{fold}} : k = 0, \dots, 2^R - \iota\right\},$$

As in the case of periodic wavelets, we can decompose T into three subspaces containing interior and boundary wavelets respectively. As before, suppose that $\mathrm{supp}(\phi) \subseteq [-p+1, p]$, $p \in \mathbb{N}$. Since $f(x) = f^{\mathrm{fold}}(x)$ for $x \in [0,1]$ whenever $\mathrm{supp}(f) \subseteq [0,1]$, we have

$$\mathrm{T} = \mathrm{T}^{\mathrm{left}} \oplus \mathrm{T}^{i} \oplus \mathrm{T}^{\mathrm{right}},$$

where

$$\mathrm{T}^{i} = \left\{\phi_{R,k} : k = p, \dots, 2^R - p - 1\right\},$$

and

$$\mathrm{T}^{\mathrm{left}} = \left\{\phi_{R,k}^{\mathrm{fold}} \mathbb{I}_{[0,1]} : k = 0, \dots, p-1\right\}, \quad \mathrm{T}^{\mathrm{right}} = \left\{\phi_{R,k}^{\mathrm{fold}} \mathbb{I}_{[0,1]} : k = 2^R - p, \dots, 2^R - \iota\right\}.$$

5.1.3 Boundary wavelets

Unfortunately, folded wavelets only retain one vanishing moment, and consequently do not lead to high approximation orders for smooth functions. To get such orders, one may follow the boundary wavelet construction of Cohen, Daubechies & Vial [16]. Let $p \in \mathbb{N}$ be given and denote the corresponding scaling and wavelet functions by ϕ and ψ. Note that the support of these functions is contained in $[-p+1, p]$. We define a new basis on $[0,1]$ as follows. We set

$$\phi^{\text{int}}_{j,k}(x) = \begin{cases} 2^{j/2}\phi(2^j x - k) & p \leq k < 2^j - p \\ 2^{j/2}\phi^{\text{left}}_k(2^j x) & 0 \leq k < p \\ 2^{j/2}\phi^{\text{right}}_{2^j-k-1}(2^j(x-1)) & 2^j - p \leq k < 2^j, \end{cases} \tag{5.7}$$

and similarly for the wavelet functions $\psi^{\text{int}}_{j,k}$. Here the functions ϕ^{left}_k and ϕ^{right}_k are particular boundary scaling functions. See [16] for details. We may now define an MRA

$$V^{\text{int}}_j = \text{span}\left\{\phi^{\text{int}}_{j,k} : \ k = 0, \ldots, 2^j - 1\right\}, \quad W^{\text{int}}_j = \text{span}\left\{\phi^{\text{int}}_{j,k} : \ k = 0, \ldots, 2^j - 1\right\},$$

which, for $J \geq \log_2(2p)$ gives the reconstruction space

$$\mathrm{T} = V^{\text{int}}_J \oplus W^{\text{int}}_J \oplus \cdots \oplus W^{\text{int}}_{R-1} = V^{\text{int}}_R. \tag{5.8}$$

Note that, as before, we may decompose

$$\mathrm{T} = \mathrm{T}^{\text{left}} \oplus \mathrm{T}^i \oplus \mathrm{T}^{\text{right}},$$

where

$$\mathrm{T}^i = \text{span}\left\{\phi_{R,k} : k = p, \ldots, 2^R - p - 1\right\},$$

contains the unmodified scaling functions with support in $[0,1]$ and

$$\mathrm{T}^{\text{left}} = \text{span}\left\{\phi^{\text{int}}_{R,k}\mathbb{I}_{[0,1]} : k = 0, \ldots, p-1\right\}, \qquad \mathrm{T}^{\text{right}} = \text{span}\left\{\phi^{\text{int}}_{R,k}\mathbb{I}_{[0,1]} : k = 2^R - p, \ldots, 2^R - 1\right\}.$$

Note that the wavelets described above are particularly well suited for smooth functions. Indeed, if $f \in \mathrm{H}^s(0,1)$, where $\mathrm{H}^s(0,1)$ denotes the usual Sobolev space and $0 \leq s < p$, then the error

$$\|f - \mathcal{P}_{\mathrm{T}} f\| = \mathcal{O}\left(2^{-sR}\right), \quad R \to \infty, \tag{5.9}$$

where T is the subspace (5.8). Since NUGS is quasi-optimal, we therefore obtain exactly the same approximation rates when reconstructing f from nonuniform Fourier samples, provided the bandwidth K (or N in the frame case) is chosen suitably large. Our main results below establish that K (or N) need only scale linearly in $M = 2^R$ to guarantee this.

Remark 5.3 Note that the wavelets introduced in this section – namely, periodic, folded or boundary – are considered as functions with support contained in $[0,1]$, even though they are actually defined over $\mathbb{R}$. In particular, their Fourier transforms are taken as integrals over $[0,1]$, as opposed to $\mathbb{R}$. Conversely, the scaling function ϕ is defined over the whole of $\mathbb{R}$, and thus its Fourier transform is also taken over $\mathbb{R}$.

5.2 Main results

5.2.1 General wavelets

We commence with the (K, δ)-dense case:

Theorem 5.4. *Let Ω be a (K, δ)-dense sampling scheme and suppose that T is the reconstruction space (5.3) of dimension 2^R generated by the first 2^R elements of a periodic wavelet basis (see §5.1.1). Suppose that either of the following conditions holds:*

(i) the scaling function $\phi \in \mathrm{L}^2(\mathbb{R})$ and $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ forms an orthonormal basis of V_0,

(ii) the scaling function ϕ satisfies

$$|\hat{\phi}(\omega)| \leq \frac{c}{(1+|\omega|)^{\alpha}}, \quad \omega \in \mathbb{R}, \tag{5.10}$$

for some $\alpha > \frac{1}{2}$, and the system $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ forms a Riesz basis of V_0.

Then for any $0 < \epsilon < 1-\delta$ there exists a $c_0 = c_0(\epsilon)$ such that if $K \geq c_0(\epsilon)2^R$ then the reconstruction constant

$$C(\Omega, \mathrm{T}) \leq \frac{1+\delta}{1-\delta-\epsilon}.$$

Theorem 5.5. *Let Ω be a (K, δ)-dense sampling scheme and suppose that either: (i) T is the reconstruction space (5.6) of dimension 2^R generated by the first 2^R elements of the folded wavelets basis of §5.1.2, or (ii) T is the reconstruction space (5.8) of dimension 2^R generated by the first 2^R elements of the folded wavelets basis of §5.1.3. Suppose that $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a Riesz basis for V_0 and that ϕ satisfies (5.10) for some $\alpha > \frac{1}{2}$. Then given $0 < \epsilon < 1 - \delta$ there exists a $c_0 = c_0(\epsilon)$ such that*

$$C(\Omega, \mathrm{T}) \leq \frac{1+\delta}{1-\delta-\epsilon}, \qquad K \geq c_0(\epsilon)2^R.$$

These theorems state that the bandwidth K needs to scale linearly with the dimension of the reconstruction space T in the case of wavelets. Note that the smoothness assumption (5.10) is extremely mild. For example, it holds if $\phi \in \mathrm{H}^{\alpha}(\mathbb{R})$ for $\alpha > \frac{1}{2}$, and consequently includes all cases of interest in practice. We remark also that the stipulation of a Riesz basis in these theorems is not necessary since this is implied by the MRA property. It is included merely for clarity.

We now consider the frame case. To this end, let $\{\omega_n\}_{n\in\mathbb{Z}}$ be a sequence such that the set $\{\mathrm{e}^{2\mathrm{i}\pi\omega_n\cdot}\mathbb{I}_{[0,1]}(\cdot)\}_{n\in\mathbb{Z}}$ forms a Fourier frame for H. For simplicity, we assume that the ω_n are nondecreasing.

Theorem 5.6. *Let $\Omega = \{\omega_n : |n| \leq N\}$, where $\{\omega_n : n \in \mathbb{Z}\}$ is a nondecreasing sequence that gives rise to a Fourier frame with frame bounds A and B. Let T be the reconstruction space of dimension 2^R consisting of either periodic (§5.1.1), folded (§5.1.2) or boundary wavelets (§5.1.3), and suppose that ϕ satisfies (5.10) for some $\alpha > \frac{1}{2}$. Then given $0 < \epsilon < A$ there exists a $c_0 = c_0(\epsilon)$ such that*

$$C(\Omega, \mathrm{T}) \leq \sqrt{\frac{B}{A-\epsilon}}, \quad \forall N \geq c_0(\epsilon)2^R.$$

These results, in combination with (5.9) imply the following important corollary for NUGS with boundary wavelets:

Corollary 5.7. *Let* T *be the reconstruction space consisting of the boundary wavelets of* §*5.1.3. If* $f \in \mathrm{H}^s(0,1)$*, where* $0 \le s < p$*, let* $\tilde{f}$ *denote the NUGS reconstruction based on a sampling scheme* Ω*. Then* $\|f - \tilde{f}\| = \mathcal{O}\left(K^{-s}\right)$ *if* Ω *is as in Theorem 5.5 and* $\|f - \tilde{f}\| = \mathcal{O}\left(N^{-s}\right)$ *when* Ω *is as in Theorem 5.6.*

This result illustrates a critically important facet of NUGS. Namely, up to constant factors, it obtains optimal convergence rates in terms of the sampling bandwidth when reconstructing smooth functions with boundary wavelets.

5.2.2 Explicit estimates for Haar wavelets and digital signal models

The main theorems above, Theorems 5.4–5.6, do not give explicit bounds for the constant $C(\Omega,\mathrm{T})$. In general, getting explicit bounds is difficult, due primarily to the contributions of the boundary subspaces T^{left} and $\mathrm{T}^{\text{right}}$. However, for the important case of Haar wavelets, there are no such terms, and this means that explicit bounds are possible.

An important motivation for studying the Haar wavelet case is that it corresponds to the situation of a digital model for the signal f. Specifically, the reconstruction space for Haar wavelets

$$\mathrm{T} = \operatorname{span}\left\{\phi \cup \{\psi_{j,k} : k = 0,\dots,2^j-1,\ j = 0,\dots,R-1\}\right\},$$

is a special case corresponding to $M = 2^R$ of reconstruction space

$$\mathrm{U} = \mathrm{U}_M = \left\{g \in \mathrm{L}^2(0,1) : g|_{[m/M,(m+1)/M)} = \text{constant},\ m = 0,\dots,M-1\right\}, \quad M \in \mathbb{N}, \tag{5.11}$$

consisting of piecewise constant functions (i.e. digital signals where $1/M$ is the pixel size). Note that

$$\mathrm{U}_M = \operatorname{span}\left\{\sqrt{M}\phi(M\cdot - m) : m = 0,\dots,M-1\right\}, \tag{5.12}$$

is a subspace generated by shifts of the pixel indicator function $\phi(x) = \mathbb{I}_{[0,1]}(x)$. This digital signal model is popular in imaging. In particular, it is the basis of the widely-used the fast, iterative reconstruction technique for MRI [38].

Our next result gives an explicit upper bound for the reconstruction constant in this case:

Theorem 5.8. *Let* Ω *be a* (K,δ)*-dense sampling scheme and let* $\mathrm{T} \subseteq \mathrm{U}_M$*, where* U_M *is given by (5.12) for* $\phi(x) = \mathbb{I}_{[0,1]}(x)$ *and* $M \le 2K$*. Then the following hold:*

(i) If $2K/M \in \mathbb{N}$ *then*

$$C(\Omega,\mathrm{T}) \le \frac{\pi}{2}\left(\frac{1+\delta}{1-\delta}\right).$$

(ii) If $2K/M \notin \mathbb{N}$ *and* $M \ge 2$ *then*

$$C(\Omega,\mathrm{T}) \le c_0\left(\frac{1+\delta}{1-\delta}\right), \quad c_0 = \frac{1}{\operatorname{sinc}\left(\pi/2 + \pi\delta/M\right)}.$$

In particular, $c_0 \sim \pi/2$ *for* $M \gg 1$*.*

This theorem demonstrates that reconstruction constant is always mild whenever M is at most $2K$. Note that the iterative reconstruction technique [38] is a specific instance of NUGS, where the term ‘iterative’ refers to the use of conjugate gradient-type algorithms to solve the least squares problem (recall the discussion in §3.4). Thus, Theorem 5.8 provides an explicit guarantee for stable, quasi-optimal reconstruction with the iterative method. We note, however, that NUGS allows for reconstructions in arbitrary subspaces. In particular, spaces T that are better suited for the signal f to be recovered, such as those consisting of the higher-order, boundary wavelets discussed in §5.1.3, for example.

Remark 5.9 Theorem 5.8 can be easily generalized to the case where $\phi \in \mathrm{L}^2(\mathbb{R})$ is an arbitrary kernel such that (i) $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ forms a Riesz basis and (ii) $\mathrm{U}_M \subseteq \mathrm{H}$. Note that (ii) means that none of the shifted versions $\sqrt{M}\phi(M \cdot -m)$ can overlap with the interval endpoints $x = 0$ and $x = 1$. Thus such spaces have poor approximation properties for functions that do not themselves vanish at the endpoints. In such cases, it is preferable to consider the interval wavelet constructions based on periodic, folded or boundary wavelets, as described in the previous section, and whose reconstruction constants are addressed by Theorems 5.4 and 5.5 (albeit without explicit bounds).

5.3 Proofs

We first require the following technical lemma:

Lemma 5.10. *Let $I \subseteq \mathbb{N}$ be a finite index set and suppose that $\{\varphi_n : n \in I\} \subseteq \mathrm{H}$ is a Reisz basis for its span $\mathrm{T} = \mathrm{span}\{\varphi_n : n \in I\}$ with constants d_1 and d_2. Let I be partitioned into disjoint subsets $I_1, \ldots, I_r$, and write $\mathrm{T}_i = \mathrm{span}\{\varphi_n : n \in I_i\}$. Let $E(\mathrm{T}, z)$ and $\tilde{E}(\mathrm{T}, N)$ be given by (4.4) and (4.9) respectively. Then*

$$E(\mathrm{T}, z) \leq \sqrt{\frac{d_2}{d_1}\sum_{i=1}^{r} E(\mathrm{T}_i, z)^2}, \qquad \tilde{E}(\mathrm{T}, N) \leq \sqrt{\frac{d_2}{d_1}\sum_{i=1}^{r} \tilde{E}(\mathrm{T}_i, N)^2}$$

Proof. Let $f = \sum_{n\in I} \alpha_n \varphi_n \in \mathrm{T}\backslash\{0\}$ and write

$$f = \sum_{i=1}^{r} f_i, \qquad f_i = \sum_{n\in I_i} \alpha_n \varphi_n.$$

Note that

$$\|\hat{f}\|^2_{\mathbb{R}\backslash(-z,z)} \leq \left(\sum_{i=1}^{r} \|\widehat{f}_i\|_{\mathbb{R}\backslash(-z,z)}\right)^2 \leq \left(\sum_{i=1}^{r} E(\mathrm{T}_i, z)\|f_i\|\right)^2 \leq \sum_{i=1}^{r} E(\mathrm{T}_i, z)^2 \sum_{i=1}^{r} \|f_i\|^2.$$

Also, since $\{\varphi_n\}_{n\in I}$ forms a Riesz basis, we have $\sum_{i=1}^{r} \|f_i\|^2 \leq d_2/d_1 \|f\|^2$. Therefore

$$\frac{\|\hat{f}\|^2_{\mathbb{R}\backslash(-z,z)}}{\|f\|^2} \leq \frac{d_2}{d_1}\sum_{i=1}^{r} E(\mathrm{T}_i, z)^2.$$

Taking the supremum over f now gives the result for $E(\mathrm{T}, z)$. For $\tilde{E}(\mathrm{T}, N)$, we first note that $\sum_{|n|>N} |\hat{f}_i(\omega_n)|^2 < \infty$, $i = 1, \ldots, r$, since $\{\omega_n\}_{n\in\mathbb{Z}}$ gives rise to the Fourier frame and $f_i \in \mathrm{L}^2(0, 1)$, $i = 1, \ldots, r$. Therefore, we can apply Minkowski's inequality to get

$$\sqrt{\sum_{|n|>N} |\hat{f}(\omega_n)|^2} \leq \sum_{i=1}^{r} \sqrt{\sum_{|n|>N} |\hat{f}_i(\omega_n)|^2}.$$

Thus,

$$\sum_{|n|>N} |\hat{f}(\omega_n)|^2 \leq \left(\sum_{i=1}^{r} \tilde{E}(\mathrm{T}^i, N)\|f_i\|\right)^2 \leq \frac{d_2}{d_1}\|f\|^2 \sum_{i=1}^{r} \tilde{E}(\mathrm{T}^i, N)^2,$$

as required. $\square$

Recall that all the wavelet reconstruction systems introduced in the previous section can be decomposed into interior wavelets having support in $[0,1]$ and boundary wavelets that intersect the endpoints $x = 0, 1$. This lemma allows us to estimate the residuals $E(\mathrm{T}, z)$ and $\tilde{E}(\mathrm{T}, N)$ by considering each subspace separately.

The next two results address the interior wavelets:

Proposition 5.11. *Let $\phi \in \mathrm{L}^2(\mathbb{R})$ have compact support and suppose that $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ forms a Riesz basis for its span with constants d_1 and d_2. Let $M \in \mathbb{N}$, $M_1, M_2 \in \mathbb{Z}$ and*

$$\mathrm{T} = \mathrm{span}\left\{\sqrt{M}\phi(M \cdot - m) : \ m = M_1, \ldots, M_2\right\},$$

and suppose that M, M_1, M_2 are such that $\mathrm{T} \subseteq \mathrm{H}$. Then the following hold:

1. *Given $\epsilon > 0$ there exists a $c_0 = c_0(\epsilon)$ such that*

$$E(\mathrm{T}, z)^2 < 1 - \frac{d_1}{d_2} + \epsilon, \quad z \geq c_0 M.$$

2. *Suppose that ϕ satisfies (5.10) for some $\alpha > \frac{1}{2}$. Then there exists a $c_0 = c_0(\epsilon)$ such that*

$$E(\mathrm{T}, z)^2 < \epsilon, \quad z \geq c_0 M.$$

Proof. Let $f \in \mathrm{T}$ and write

$$f(x) = \sqrt{M} \sum_{k=M_1}^{M_2} a_k \phi(Mx - k).$$

Since $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a Riesz basis, we find that

$$d_1 \sum_{k=M_1}^{M_2} |a_k|^2 \leq \|f\|^2 \leq d_2 \sum_{k=M_1}^{M_2} |a_k|^2. \tag{5.13}$$

Moreover, a simple calculation gives that

$$\hat{f}(\omega) = \frac{1}{\sqrt{M}} \hat{\phi}\left(\frac{\omega}{M}\right) \Psi\left(\frac{\omega}{M}\right), \quad \omega \in \mathbb{R}, \tag{5.14}$$

where $\Psi(x) = \sum_{k=M_1}^{M_2} a_k \mathrm{e}^{-2\pi \mathrm{i} k x}$ is a trigonometric polynomial with

$$\|\Psi\|^2 = \sum_{k=M_1}^{M_2} |a_k|^2.$$

Thus, using (5.13) we get

$$d_1 \|\Psi\|^2 \leq \|f\|^2 \leq d_2 \|\Psi\|^2. \tag{5.15}$$

We now estimate $\|\hat{f}\|^2_{(-z,z)}$. By (5.14), we have

$$\|\hat{f}\|^2_{(-z,z)} = \frac{1}{M} \int_{|\omega|<z} |\hat{\phi}(\omega/M)|^2 \, |\Psi(\omega/M)|^2 \, \mathrm{d}\omega = \int_{|t|<z/M} |\hat{\phi}(t)|^2 \, |\Psi(t)|^2 \, \mathrm{d}t.$$

Suppose that $z \geq M$ and write $\lfloor z/M \rfloor = n_0 + 1$, where $n_0 \in \mathbb{N}_0$. Then, since Ψ is 1-periodic,

$$
\begin{aligned}
\|\hat{f}\|^2_{(-z,z)} &\geq \int_{t=-n_0}^{n_0+1} |\hat{\phi}(t)|^2 \, |\Psi(t)|^2 \, \mathrm{d}t \\
&= \sum_{|n| \leq n_0} \int_0^1 |\hat{\phi}(t+n)|^2 \, |\Psi(t+n)|^2 \, \mathrm{d}t \\
&\geq \left(\min_{t \in [0,1]} \sum_{|n| \leq n_0} |\hat{\phi}(n+t)|^2 \right) \int_0^1 |\Psi(t)|^2 \, \mathrm{d}t \\
&\geq \frac{1}{d_2} \left(\min_{t \in [0,1]} \sum_{|n| \leq n_0} |\hat{\phi}(n+t)|^2 \right) \|f\|^2, \qquad (5.16)
\end{aligned}
$$

where the final inequality follows from (5.15). By [5, Lem. 5.4], there exists an $n_0 \in \mathbb{N}$ sufficiently large such that the term in brackets is greater than $d_1 - \epsilon d_2$. Thus we get

$$\|\hat{f}\|^2_{(-z,z)} \geq \left(\frac{d_1}{d_2} - \epsilon \right) \|f\|^2.$$

We now use the definition of $E(\mathrm{T}, z)^2$ to complete part 1. of the proof.

Our approach for part 2. will be similar, but we shall estimate the tail $\|\hat{f}\|^2_{\mathbb{R}\setminus(-z,z)}$. Repeating the steps of the above proof, we find that

$$\|\hat{f}\|^2_{\mathbb{R}\setminus(-z,z)} \leq \frac{1}{d_1} \left(\sup_{t \in [0,1]} \sum_{|n| \geq n_0} |\hat{\phi}(n+t)|^2 \right) \|f\|^2.$$

Using the smoothness assumption (5.10), we find that

$$\sup_{t \in [0,1]} \sum_{|n| \geq n_0} |\hat{\phi}(n+t)|^2 \lesssim (n_0)^{1-2\alpha}.$$

Hence, if $z \geq c_0(\epsilon) M$ for some c_0, then we have

$$\|\hat{f}\|^2_{\mathbb{R}\setminus(-z,z)} \leq \epsilon \|f\|^2,$$

from which the result follows. ☐

Having addressed the case of (K, δ)-dense samples, we now consider frame samples. Recalling the setup of §4.2, let $\{\omega_n : n \in \mathbb{Z}\}$ be a nondecreasing sequence giving rise to a Fourier frame. Set $\Omega_N = \{\omega_n : |n| \leq N\}$, and suppose that $\mathcal{S}_N$ is given by (4.7).

Proposition 5.12. *Let $\{\omega_n\}_{n \in \mathbb{Z}} \subseteq \mathbb{R}$ be a nondecreasing sequence of frequencies that rise to a Fourier frame for* H, *and suppose that ϕ and* T *are as in Proposition 5.11. If ϕ satisfies (5.10) for some $\alpha > \frac{1}{2}$, then given $\epsilon > 0$ there exists a $c_0 = c_0(\epsilon)$ such that*

$$\tilde{E}(\mathrm{T}, N) < \epsilon, \quad \forall N \geq c_0 M.$$

We first require the following two lemmas:

Lemma 5.13. *Let $\{\omega_n\}_{n\in\mathbb{Z}}$ be an increasing sequence of separated points with minimal separation $\eta = \inf_{n\in\mathbb{Z}}\{\omega_{n+1}-\omega_n\} > 0$. Then there exists a set of points $\{\tilde{\omega}_n\}_{n\in\mathbb{Z}}$ with minimal separation at least $\eta/2$ such that $\{\omega_n\}_{n\in\mathbb{Z}} \subseteq \{\tilde{\omega}_n\}_{n\in\mathbb{Z}}$ and*

$$\sup_{n\in\mathbb{Z}}\{\tilde{\omega}_{n+1}-\tilde{\omega}_n\} \leq \eta.$$

Proof. Let $n \in \mathbb{Z}$. If $\omega_{n+1}-\omega_n = \eta$ then we do nothing. Otherwise, let $k \in \mathbb{N}$ be the smallest integer such that $\omega_{n+1}-\omega_n \leq (k+1)\eta$. Introduce the new points

$$\omega_n + r\eta, \quad r = 1,\dots,k-1,$$

as well as

$$\frac{1}{2}\left(\omega_n + (k-1)\eta + \omega_{n+1}\right).$$

These new points are at least $\eta/2$ separated, and have maximal separation at most η. □

Lemma 5.14. *Let $x_0 \leq x_1 < x_2 < \ldots < x_N \leq x_{N+1}$ where $N \in \mathbb{N}\cup\{\infty\}$, and suppose that $\delta = \max_{n=0,\dots,N}\{x_{n+1}-x_n\} < \infty$. Let $f \in \mathrm{H}^1(a,b)$, where $a = \frac{1}{2}(x_1+x_0)$, $b = \frac{1}{2}(x_{N+1}+x_N)$ and $\mathrm{H}^1(a,b)$ denotes the standard Sobolev space of first order on the interval (a,b). If $\mu_n = \frac{1}{2}(x_{n+1}-x_{n-1})$, $n = 1,\dots,N$, then the following inequalities hold:*

$$\left(\|f\|_{[a,b]} - \frac{\delta}{\pi}\|f'\|_{[a,b]}\right)^2 \leq \sum_{n=1}^{N}\mu_n|f(x_n)|^2 \leq \left(\|f\|_{[a,b]} + \frac{\delta}{\pi}\|f'\|_{[a,b]}\right)^2.$$

Proof. The proof of this lemma is similar to that of Lemma 4.2. Let $z_n = \frac{1}{2}(x_n + x_{n-1})$ and define $\chi(x) = \sum_{n=1}^{N} f(x_n)\mathbb{I}_{[z_n,z_{n+1})}(x)$. Note that $z_1 = a$, $z_{N+1} = b$ and that

$$\sum_{n=1}^{N}\mu_n|f(x_n)|^2 = \|\chi\|^2_{[a,b]}.$$

We now have

$$\|f-\chi\|^2_{[a,b]} = \sum_{n=1}^{N}\int_{z_n}^{z_{n+1}} |f(x)-f(x_n)|^2 \,\mathrm{d}x,$$

and after an application of Wirtinger's inequality, we obtain

$$\|f-\chi\|^2_{[a,b]} \leq \frac{\delta^2}{\pi^2}\|f'\|^2_{[a,b]},$$

which gives the result. □

Proof of Proposition 5.12. Recall from Theorem 4.6 that any sequence $\{\omega_n\}_{n\in\mathbb{Z}}$ that gives a frame is necessarily relatively separated, i.e. it is a finite union of separated sequences. Since we wish to obtain an upper bound for

$$\sum_{|n|>N}|\hat{f}(\omega_n)|^2,$$

for any $f \in \mathrm{T}$, we may therefore assume without loss of generality that $\{\omega_n\}_{n\in\mathbb{Z}}$ is a separated sequence with separation η. Moreover, after an application of Lemma 5.13, we may assume without loss of generality that $\{\omega_n\}_{n\in\mathbb{Z}}$ is $\eta/2$ separated with maximal spacing at most η.

As in the proof of Proposition 5.11 let $f = \sum_{k=M_1}^{M_2} a_k \sqrt{M}\phi(M\cdot -k) \in \mathrm{T}$ and write $\tilde{\Psi}(x) = \sum_{k=M_1}^{M_2} a_k \mathrm{e}^{-2\pi \mathrm{i} kx}$ so that

$$\hat{f}(\omega) = \frac{1}{\sqrt{M}} \hat{\phi}\left(\frac{\omega}{M}\right) \tilde{\Psi}\left(\frac{\omega}{M}\right). \tag{5.17}$$

Let

$$\Psi(x) = \mathrm{e}^{2\pi \mathrm{i} M_3 x} \tilde{\Psi}(x) = \sum_{k=M_1-M_3}^{M_2-M_3} a_{k+M_3} \mathrm{e}^{-2\pi \mathrm{i} kx}, \qquad M_3 = \left\lceil \frac{M_1+M_2}{2} \right\rceil, \tag{5.18}$$

so that $|\Psi(x)| = |\tilde{\Psi}(x)|$. By (5.17) we also have $|\hat{f}(\omega)| = \frac{1}{\sqrt{M}} |\hat{\phi}(\omega/M)||\tilde{\Psi}(\omega/M)| = \frac{1}{\sqrt{M}} |\hat{\phi}(\omega/M)||\Psi(\omega/M)|$, and therefore

$$\begin{aligned} \sum_{n>N} |\hat{f}(\omega_n)|^2 &\leq \frac{1}{M} \sum_{n>N} |\hat{\phi}(\omega_n/M)|^2 |\Psi(\omega_n/M)|^2 \\ &\leq \frac{1}{M} \sum_{l=0}^{\infty} \sup_{\omega \in I_l} |\hat{\phi}(\omega/M)|^2 \sum_{n:\omega_n \in I_l} |\Psi(\omega_n/M)|^2, \end{aligned}$$

where $I_l = [\omega_N + lM, \omega_N + (l+1)M)$. Since $\{\omega_n\}_{n\in\mathbb{Z}}$ is separated and increasing, we must have that $\omega_N \gtrsim N$ as $N \to \infty$. In particular $\omega_N > 0$ for sufficiently large N. By the assumption on ϕ, we therefore obtain

$$\sum_{n>N} |\hat{f}(\omega_n)|^2 \lesssim M^{2\alpha-1} \sum_{l=0}^{\infty} (\omega_N + 2lM)^{-2\alpha} \sum_{n:\omega_n \in I_l} |\Psi(\omega_n/M)|^2.$$

We now claim that the result follows, provided

$$\sum_{n:\omega_n \in I_l} |\Psi(\omega_n/M)|^2 \leq cM \|\Psi\|^2, \quad \forall l = 0, 1, 2, \ldots. \tag{5.19}$$

We shall prove that (5.19) holds in a moment. First, however, let us show how (5.19) implies the result. Substituting this bound into the previous expression gives

$$\sum_{n>N} |\hat{f}(\omega_n)|^2 \lesssim M^{2\alpha} \sum_{l=0}^{\infty} (\omega_N + 2lM)^{-2\alpha} \|\Psi\|^2 \lesssim \left(\frac{\omega_N}{M}\right)^{1-2\alpha} \|\Psi\|^2.$$

Similarly, we also get

$$\sum_{n<-N} |\hat{f}(\omega_n)|^2 \lesssim \left(\frac{|\omega_{-N}|}{M}\right)^{1-2\alpha} \|\Psi\|^2.$$

An application of (5.15) now gives

$$\tilde{E}(\mathrm{T}, N)^2 \lesssim \frac{1}{d_1} \left(\frac{\min\{\omega_N, |\omega_{-N}|\}}{M}\right)^{1-2\alpha}.$$

Since $\omega_N, |\omega_{-N}| \gtrsim N$ as $N \to \infty$, the result now follows.

It remains to establish (5.19). Write $\{\omega_n/M : \omega_n \in I_l\} = \{x_1, \ldots, x_L\}$ where

$$\omega_N/M + l \leq x_1 < x_2 < \ldots < x_L \leq \omega_N/M + l + 1,$$

and set $x_0 = x_1$ and $x_{L+1} = x_L$. Note that $\eta/(2M) \le x_{n+1} - x_n \le \eta/M$. Therefore

$$\sum_{n:\omega_n \in I_l} |\Psi(\omega_n/M)|^2 = \sum_{n=1}^{L} |\Psi(x_n)|^2 \le \frac{2M}{\eta} \sum_{n=1}^{L} \mu_n |\Psi(x_n)|^2,$$

where $\mu_n = \frac{1}{2}(x_{n+1} - x_{n-1})$. Hence, by Lemma 5.14 we have

$$\sum_{n:\omega_n \in I_l} |\Psi(\omega_n/M)|^2 \le \frac{2M}{\eta} \left[\|\Psi\|_{[a,b]} + \frac{\eta}{M\pi} \|\Psi'\|_{[a,b]} \right]^2,$$

where $a = \frac{1}{2}(x_1 + x_0) = x_1$ and $b = \frac{1}{2}(x_{L+1} + x_L) = x_L$. Note that $|b - a| \le 1$. Hence since Ψ is periodic, we get

$$\sum_{n:\omega_n \in I_l} |\Psi(\omega_n/M)|^2 \le \frac{2M}{\eta} \left[\|\Psi\| + \frac{\eta}{M\pi} \|\Psi'\| \right]^2.$$

To prove the result, we only need to show that $\|\Psi'\| \le M\pi \|\Psi\|$. Since Ψ is a trigonometric polynomial given by (5.18), we have

$$\|\Psi'\| \le 2 \max\{M_2 - M_3, M_3 - M_1\} \pi \|\Psi\|.$$

Thus it remains to show that $M_2 - M_3, M_3 - M_1 \le M/2$. Since $\mathrm{T} \subseteq \mathrm{H}$ by assumption, the function ϕ must have compact support. Let $\operatorname{supp}(\phi) \subseteq [a, b]$. Then we must also have that $-a \le M_1 \le M_2 \le M - b$. In particular, $M_2 - M_1 \le M - (b - a) < M$. Therefore

$$M_2 - M_3 \le M_2 - \frac{M_1 + M_2}{2} < \frac{M}{2}, \quad M_3 - M_1 \le \frac{M_1 + M_2}{2} + 1 - M_1 \le \frac{M}{2} + 1 - \frac{b - a}{2}.$$

Since $M_3 - M_1 \in \mathbb{N}$ and $b - a > 0$ we obtain the result. $\square$

We are now in a position to prove Theorems 5.4 and 5.5.

Proof of Theorems 5.4 and 5.5. By Theorem 4.5, it suffices to consider $E(\mathrm{T}, z)$. Recall that in all three cases – periodic, folded or boundary wavelets – the reconstruction space T can be decomposed as $\mathrm{T} = \mathrm{T}^{\text{left}} \oplus \mathrm{T}^{i} \oplus \mathrm{T}^{\text{right}}$. Lemma 5.10 now gives

$$E(\mathrm{T}, z)^2 \le \frac{d_2}{d_1} \left(E(\mathrm{T}^{\text{left}}, z)^2 + E(\mathrm{T}^{i}, z)^2 + E(\mathrm{T}^{\text{right}}, z)^2 \right).$$

The subspace T^i contains wavelets supported in $[0, 1]$, an application of Proposition 5.11 gives $E(\mathrm{T}^i, z)^2 < \epsilon$ in both case (i) and case (ii) of Theorem 5.4 (recall in case (i) that $\{\phi(\cdot - k)\}_{k \in \mathbb{Z}}$ is an orthonormal basis, and therefore $d_1 = d_2 = 1$), as well as in Theorem 5.5. Thus it remains to show in all cases that $E(\mathrm{T}^{\text{left}}, z)$ and $E(\mathrm{T}^{\text{right}}, z)$ can be made arbitrarily small with $z \gtrsim 2^R$

Consider the subspace T^{left} (the case of $\mathrm{T}^{\text{right}}$ is identical). For all three wavelet constructions, we may write

$$\mathrm{T}^{\text{left}} = \operatorname{span}\left\{ \Phi_{R,k} \mathbb{I}_{[0,1]} : k = 0, \ldots, p - 1 \right\},$$

where $\Phi_{R,k}$ is either $\phi^{\text{per}}_{R,k}$ (periodic), $\phi^{\text{fold}}_{R,k}$ (folded) or $\phi^{\text{int}}_{R,k}$ (boundary). The functions $\Phi_{R,k}\mathbb{I}_{[0,1]}$ form a Riesz basis for T^{left} with bounds d_1 and d_2. Hence, if

$$f = \sum_{k=0}^{p-1} \alpha_k \Phi_{R,k} \mathbb{I}_{[0,1]} \in \mathrm{T}^{\text{left}},$$

then

$$d_1 \sum_{k=0}^{p-1} |\alpha_k|^2 \le \|f\|^2 \le d_2 \sum_{k=0}^{p-1} |\alpha_k|^2.$$

Now consider $\|\hat{f}\|_{\mathbb{R}\setminus(-z,z)}$. By the Cauchy–Schwarz inequality and the above inequality,

$$\|\hat{f}\|_{\mathbb{R}\setminus(-z,z)} \le \sum_{k=0}^{p-1} |\alpha_k| \|(\Phi_{R,k}\mathbb{I}_{[0,1]})^\wedge\|_{\mathbb{R}\setminus(-z,z)} \le \sqrt{p/d_1}\|f\| \max_{0\le k\le p-1} \left\{\|(\Phi_{R,k}\mathbb{I}_{[0,1]})^\wedge\|_{\mathbb{R}\setminus(-z,z)}\right\},$$

Thus, to complete the proof, we only need to show that there exists a $c_0 = c_0(\epsilon)$ such that

$$\|(\Phi_{R,k}\mathbb{I}_{[0,1]})^\wedge\|_{\mathbb{R}\setminus(-z,z)} < \epsilon, \qquad \forall k = 0, \dots, p-1, \tag{5.20}$$

whenever $z \ge c_0(\epsilon)2^R$.

Assume now that $2^{R-1} > p$. Then one can determine the following:

(a) For periodic wavelets, $\Phi_{R,k}(x) = \phi_{R,k}(x) + \phi_{R,k}(x-1)$.

(b) For folded wavelets, $\Phi_{R,k}(x) = \phi_{R,k}(x) + \phi_{R,k}(-x)$.

(c) For boundary wavelets, $\Phi_{R,k}(x)$ can be written as a finite linear combination of the functions $\phi_{R,k}(x)$, where $k = -p+1, \dots, p-1$.

Note that (a) and (b) follow by first writing $\phi^{\text{per}}_{R,k}$ and $\phi^{\text{fold}}_{R,k}$ in terms of infinite sums using the periodization and folding operations (5.2) and (5.5) and then by using the fact that $\operatorname{supp}(\phi) \subseteq [-p+1, p]$. Case (c) was shown in [16]. Since in all cases $\Phi_{R,k}$ can be written as a finite sum with a number of terms independent of R, it therefore suffices to show that

$$\|(\phi_{R,k}\mathbb{I}_{[0,1]})^\wedge\|_{\mathbb{R}\setminus(-z,z)}, \|(\phi_{R,k}(\cdot-1)\mathbb{I}_{[0,1]})^\wedge\|_{\mathbb{R}\setminus(-z,z)}, \|(\phi_{R,k}(-\cdot)\mathbb{I}_{[0,1]})^\wedge\|_{\mathbb{R}\setminus(-z,z)} < \epsilon, \tag{5.21}$$

where $k = -p+1, \dots, p+1$ for the first term and $k = 0, \dots, p-1$ for the second two terms, whenever $z \ge c_0(\epsilon)2^R$. Note that

$$\left|\left(\phi_{R,k}(\cdot+l)\mathbb{I}_{[0,1]}\right)^\wedge(\omega)\right| = 2^{-R/2}\left|\int_{2^R l-k}^{2^R(l+1)-k} \phi(y)\mathrm{e}^{-2\pi \mathrm{i}\omega y/2^R}\,\mathrm{d}y\right|.$$

Suppose that $l = 0$. Then the integration interval is $[-k, 2^R - k]$. Since $\operatorname{supp}(\phi) = [-p+1, p]$, we can replace this by $[-k, p]$ to give

$$\left|\left(\phi_{R,k}(\cdot)\mathbb{I}_{[0,1]}\right)^\wedge(\omega)\right| = 2^{-R/2}\left|\widehat{\phi^{[-k,p]}}(\omega/2^R)\right|, \quad k = -p+1, \dots, p-1,$$

where, for $a < b$,

$$\phi^{[a,b]}(x) = \phi(x)\mathbb{I}_{[a,b]}(x).$$

Similarly, for $l = -1$ we have

$$\left|\left(\phi_{R,k}(\cdot-1)\mathbb{I}_{[0,1]}\right)^\wedge(\omega)\right| = 2^{-R/2}\left|\widehat{\phi^{[-p+1,k]}}(\omega/2^R)\right|, \quad k = 0, \dots, p-1.$$

Likewise

$$\left|\left(\phi_{R,k}(-\cdot)\mathbb{I}_{[0,1]}\right)^\wedge(\omega)\right| = 2^{-R/2}\left|\widehat{\phi^{[-p+1,k]}}(-\omega/2^R)\right|, \quad k = 0, \dots, p-1.$$

Thus, to establish (5.21), and therefore (5.20), it suffices to estimate the Fourier transforms of the functions $\phi^{[a,b]}$ for $(a,b) = (-k,p)$, $k = -p+1,\dots,p-1$, and $(a,b) = (-p+1,k)$, $k = 0,\dots,p-1$. We now note the following:

$$\|2^{-R/2} f(\cdot/2^R)\|_{\mathbb{R}\setminus(-z,z)} = \|f\|_{\mathbb{R}\setminus(-z/2^R, z/2^R)}, \quad f \in \mathrm{L}^2(\mathbb{R}).$$

In particular, for any fixed f,

$$\|2^{-R/2} f(\cdot/2^R)\|_{\mathbb{R}\setminus(-z,z)} < \epsilon, \tag{5.22}$$

provided $z \geq c2^R$ for appropriately large $c > 0$. Since the total number of functions $\phi^{[a,b]}$ is less than $2p$, and hence bounded independently of R, we obtain (5.21) and therefore (5.20). □

Proof of Theorem 5.6. By Theorem 4.7, we may consider $\tilde{E}(\mathrm{T}, N)$. Proceeding in a similar manner to the previous proof, we see from Lemma 5.10 that it suffices to estimate $\tilde{E}(\mathrm{T}^i, N)$, $\tilde{E}(\mathrm{T}^{\text{left}}, N)$ and $\tilde{E}(\mathrm{T}^{\text{right}}, N)$ separately. As before, $\tilde{E}(\mathrm{T}^i, N)$ can be bounded using Proposition 5.12, and hence it remains to derive bounds for $\tilde{E}(\mathrm{T}^{\text{left}}, N)$ and $\tilde{E}(\mathrm{T}^{\text{right}}, N)$ only. If we now argue in an identical way to the previous proof, i.e. by writing the spaces T^{left} and $\mathrm{T}^{\text{right}}$ as linear combinations of the functions $\phi^{[a,b]}$ whose total number is independent of R, then we see that it suffices to show the following: for an arbitrary function $f \in \mathrm{L}^2(0,1)$,

$$2^{-R} \sum_{|n|>N} \left| \hat{f}(\omega_n/2^R) \right|^2 < \epsilon, \tag{5.23}$$

provided $N \geq c2^R$ for some $c > 0$ depending only on f (this replaces the condition (5.22) in the previous proof). Recall from the proof of Proposition 5.12 that we may assume without loss of generality that the frame sequence $\{\omega_n\}_{n\in\mathbb{Z}}$ is separated with separation at least $\eta/2$ and maximal spacing at most η. Thus the points $\{\tilde{\omega}_n\}_{n\in\mathbb{Z}}$, where $\tilde{\omega}_n = \omega_n/2^R$, have maximal spacing at most $\eta/2^R$ and we find that

$$2^{-R} \sum_{|n|>N} \left| \hat{f}(\omega_n/2^R) \right|^2 \leq \frac{2}{\eta} \sum_{|n|>N} \mu_n |\hat{f}(\tilde{\omega}_n)|^2,$$

where $\mu_n = \frac{\tilde{\omega}_{n+1} - \tilde{\omega}_{n-1}}{2}$. Since $f \in \mathrm{H}$ we may apply Lemma 5.14 to get

$$2^{-R} \sum_{|n|>N} \left| \hat{f}(\omega_n/2^R) \right|^2 \leq \frac{2}{\eta} \left[\left(\|\hat{f}\|_{J_+} + \frac{\eta}{2^R\pi} \|\hat{f}'\|_{J_+} \right)^2 + \left(\|\hat{f}\|_{J_-} + \frac{\eta}{2^R\pi} \|\hat{f}'\|_{J_-} \right)^2 \right],$$

where $J_+ = (\tilde{\omega}_N, \infty)$ and $J_- = (-\infty, \tilde{\omega}_{-N})$. To obtain (5.23) we merely note that $\hat{f}' = \widehat{f_1} \in \mathrm{L}^2(\mathbb{R})$, where $f_1(x) = xf(x)$, and $\max\{\tilde{\omega}_N, -\tilde{\omega}_{-N}\} \gtrsim N/2^R$ for large N. □

Finally, we now prove Theorem 5.8:

Proof of Theorem 5.8. Since we have already shown have $C_2(\Omega) \leq (1+\delta)^2$, and since $\mathrm{T} \subseteq \mathrm{U}_M$, it is enough to estimate $C_1(\Omega, \mathrm{U}_M)$. For any $f \in \mathrm{U}_M$, we can write

$$f(x) = \sqrt{M} \sum_{m=0}^{M-1} a_m \phi(Mx - m).$$

Therefore, as before, we get

$$\hat{f}(\omega) = \frac{1}{\sqrt{M}} \hat{\phi}\left(\frac{\omega}{M}\right) \tilde{\Psi}\left(\frac{\omega}{M}\right), \tag{5.24}$$

where, for $M_0 = \lfloor M/2 \rfloor$,

$$\tilde{\Psi}(x) = \sum_{m=0}^{M-1} a_m \mathrm{e}^{-2\pi \mathrm{i} m x} = \mathrm{e}^{-2\pi \mathrm{i} M_0 x} \sum_{m=-M_0}^{M-M_0-1} a_{m+M_0} \mathrm{e}^{-2\pi \mathrm{i} m x} = \mathrm{e}^{-2\pi \mathrm{i} M_0 x} \Psi(x),$$

and $\Psi(x) = \sum_{m=-M_0}^{M-M_0-1} a_{m+M_0} \mathrm{e}^{-2\pi \mathrm{i} m x}$. Note that Ψ is a trigonometric polynomial of degree at most M_0 and moreover, since $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis, we have $\|\Psi\|^2 = \|f\|^2$. Set $x_n = \omega_n/M$ for $n = 0,\ldots,N+1$ and let $\nu_n = \frac{1}{2}(x_{n+1} - x_{n-1})$. Then we have

$$\langle \mathcal{S}f, f\rangle = \sum_{n=1}^{N} \nu_n |\Psi(x_n)|^2 |\hat{\phi}(x_n)|^2. \tag{5.25}$$

Let us first consider the case $2K/M \in \mathbb{N}$. Note that $\mathrm{U}_M \subseteq \mathrm{U}_{2K}$ in this case, and therefore it suffices to prove the result for $M = 2K$. After an application of Lemma 5.14, we obtain

$$\langle \mathcal{S}f, f\rangle \geq \min_{n=1,\ldots,N} |\hat{\phi}(x_n)|^2 \left(\|\Psi\|_{[a,b]} - \frac{\delta}{2K\pi} \|\Psi'\|_{[a,b]} \right)^2 \geq d_0 \left(\|\Psi\|_{[a,b]} - \frac{\delta}{2K\pi} \|\Psi'\|_{[a,b]} \right)^2,$$

where $a = \frac{1}{2}(x_1 + x_0) = \frac{1}{2}(x_1 + x_N) - \frac{1}{2}$, $b = \frac{1}{2}(x_N + x_{N+1}) = \frac{1}{2}(x_1 + x_N) + \frac{1}{2}$ and $d_0 = \min_{\omega\in[-1/2,1/2]} |\hat{\phi}(\omega)|^2$. Note that the second inequality here follows from the observation that $|x_n| = |\omega_n|/M \leq K/M \leq 1/2$ since the frequencies ω_n are (K,δ)-dense. Since $b - a = 1$ and Ψ is periodic, we therefore have

$$\langle \mathcal{S}f, f\rangle \geq d_0 \left(\|\Psi\| - \frac{\delta}{2K\pi} \|\Psi'\| \right)^2 \geq d_0 \left(1 - \frac{\delta M_0}{K} \right)^2 \|\Psi\|^2 \geq d_0 \left(1 - \delta\right)^2 \|\Psi\|^2,$$

where the penultimate inequality follows from $\|\Psi'\| \leq 2M_0\pi\|\Psi\|$. To complete the proof, we note that $|\hat{\phi}(\omega)| = |\mathrm{sinc}(\omega\pi)|$ and that $|\mathrm{sinc}(\omega\pi)| \geq |\mathrm{sinc}(\pi/2)| = 2/\pi$ for $\omega \in [-1/2, 1/2]$.

Now suppose that $M \leq 2K$ is arbitrary. In this case, our first step is to introduce a new subset of points $\{\tilde{x}_p\}_{p=1}^{\tilde{N}}$. We do this as follows. Let n' be the largest n such that $x_n \leq -1/2$, and let n'' be the smallest n such that $x_n \geq 1/2$. If $\tilde{N} = n'' - n' + 1$, let

$$\tilde{x}_p = x_{p+n'-1}, \quad p = 0,\ldots,\tilde{N}, \qquad \tilde{x}_{\tilde{N}+1} = 2 + x_{n'-1} + x_{n'} - x_{n''}.$$

Let $\tilde{\nu}_p = \frac{1}{2}(\tilde{x}_{p+1} - \tilde{x}_{p-1})$, and note that

$$\tilde{\nu}_p = \nu_{p+n'-1}, \quad p = 1,\ldots,\tilde{N}-1.$$

Moreover, by definition of n' and n'', we have

$$\tilde{\nu}_{\tilde{N}} = \frac{1}{2}\left(\tilde{x}_{\tilde{N}+1} - \tilde{x}_{\tilde{N}-1}\right) = \frac{1}{2}\left(2 + x_{n'-1} + x_{n'} - x_{n''} - x_{n''+1}\right) + \nu_{n''} \leq \nu_{n''}.$$

Therefore, we now obtain the following from (5.25):

$$\langle \mathcal{S}f, f\rangle \geq \min_{n=n',\ldots,n''} |\hat{\phi}(x_n)|^2 \sum_{p=1}^{\tilde{N}} \tilde{\nu}_p |\Psi(\tilde{x}_p)|^2.$$

Since the frequencies ω_n are (K,δ)-dense, we have that $x_{n'} \geq -1/2 - \delta/M$ and $x_{n''} \leq 1/2 + \delta/M$. This and an application of Lemma 5.14 now give

$$\langle \mathcal{S}f, f\rangle \geq d_0 \left(\|\Psi\|_{[a,b]} - \frac{\delta}{M\pi} \|\Psi'\|_{[a,b]} \right)^2, \qquad d_0 = \min_{\omega\in[-1/2-\delta/M,1/2+\delta/M]} |\hat{\phi}(\omega)|^2,$$

where

$$a = \frac{1}{2}\left(\tilde{x}_1 + \tilde{x}_0\right) = \frac{1}{2}\left(x_{n'} + x_{n'-1}\right),$$

and

$$b = \frac{1}{2}\left(\tilde{x}_{\tilde{N}} + \tilde{x}_{\tilde{N}+1}\right) = \frac{1}{2}\left(x_{n''} + 2 + x_{n'-1} + x_{n'} - x_{n''}\right) = a + 1.$$

Since $|b - a| = 1$, we now argue exactly as before to give the first result. $\square$

6 Bandwidth and ill-conditioning

In the previous section, we established that stable reconstruction is possible, provided the bandwidth K of the sampling scales linearly with the dimension $M = 2^R$ of the wavelet reconstruction space. We now consider the constant of this scaling:

Theorem 6.1. *Let $\Omega = \{\omega_n : n = 1, \ldots, N\} \subseteq [-K, K]$ for some $K > \frac{2}{\pi^2} + \frac{1}{2}$ and suppose that $\mathcal{S}$ is given by (3.8) with weights (4.1). Let T be the reconstruction space corresponding to either periodic (§5.1.1), folded (§5.1.2) or boundary (§5.1.3) wavelets, where $2^{R-1} > K$. Then the reconstruction constant satisfies*

$$C(\Omega, \mathrm{T}) \geq c_1/\sqrt{K} \exp\left(c_2(1-z)2^R\right),$$

where $z = \max\{\frac{1}{2}, K/2^{R-1}\}$ and $c_1, c_2 > 0$ depend only on ϕ.

This theorem, which generalizes a result proved in [5] to the case of nonuniform samples, establishes the following. Suppose that the size $M = 2^R$ of the reconstruction space is roughly $2\alpha K$. If $\alpha > 1$ then the reconstruction constant $C(\Omega, \mathrm{T})$ blows up exponentially fast as $M \to \infty$. In other words, if the bandwidth K of the sampling is not sufficiently large in comparison to the wavelet scale R, then ill-conditioning is necessarily witnessed in the reconstruction. Note that this theorem does not assume density of the samples, just that their maximal bandwidth is K. Conversely, even if $\hat{f}(\omega)$ were known for arbitrary $|\omega| \leq K$ one would still have the same result, i.e. insufficient sampling bandwidth implies ill-conditioning.

It is instructive to compare this result with Theorem 5.8, which estimates the reconstruction constant for Haar wavelets. If $M \approx 2\alpha K$ then Theorem 5.8 demonstrates that $C(\Omega, \mathrm{T})$ is bounded whenever α is less than or equal to the critical value $\alpha_0 = 1$. Conversely, if $\alpha > \alpha_0$ then exponential ill-conditioning necessarily results as a consequence of Theorem 6.1. For other wavelets, Theorems 5.4 and 5.5 show that stable reconstruction is possible for sufficiently small scaling α, but unlike the Haar wavelet case, they do not establish the exact value for α_0 that delineates the stability and instability regions.

Theorem 6.1 follows immediately from the following lemma:

Lemma 6.2. *Let Ω and $\mathcal{S}$ be as in Theorem 6.1. Let $\mathrm{T} \subseteq \mathrm{H}$ and suppose that $\mathrm{T} \supseteq \mathrm{U}$, where*

$$\mathrm{U} = \mathrm{span}\left\{\sqrt{M}\phi(M \cdot - m) : m = M_1, \ldots, M_2\right\}.$$

for some $M \in \mathbb{N}$, $M_1, M_2 \in \mathbb{Z}$ and $M > 2K$. If $\{\phi(\cdot - k)\}_{k \in \mathbb{Z}}$ is a Riesz basis for its span with bounds d_1 and d_2 then

$$C(\Omega, \mathrm{T}) \geq c_1 \sqrt{\frac{d_1}{d_2 K}} \exp\left[c_2(M_2 - M_1 - 2)(1-z)\right],$$

where $z = \max\{\frac{1}{2}, 2K/M\}$, and $c_1, c_2 > 0$ are independent of Ω, K, M, M_1, M_2 and ϕ.

Proof of Theorem 6.1. In each case, we merely set $\mathrm{U} = \mathrm{T}^i$ to be the space spanned by the interior wavelets. The result follows immediately from Lemma 6.2. $\square$

To prove Lemma 6.2, we first require the following result:

Lemma 6.3. *Let $P \in \mathbb{N}$ and $z \in (0, \frac{1}{2})$. Then there exists a constant $c > 0$ independent of P and z such that*

$$\sup \left\{ \frac{\sup_{|t|\leq 1/2} |\Psi(t)|}{\sup_{|t|\leq z} |\Psi(t)|} : \Psi(t) = \sum_{|n|\leq P} a_k \mathrm{e}^{\mathrm{i}2\pi k t},\ a_k \in \mathbb{C} \right\} \geq \exp\left(cP(1/2 - z')\right),$$

where $z' = \max\{\frac{1}{4}, z\}$.

Proof. See Proposition 6.2 in [5]. $\square$

Proof of Lemma 6.2. Note that $C(\Omega, \mathrm{T}) \geq C(\Omega, \mathrm{U})$. Let $f \in \mathrm{U}$. Arguing in the usual way, we have

$$\langle \mathcal{S} f, f \rangle = \frac{1}{M} \sum_{n=1}^{N} \mu_n |\hat{\phi}(\omega_n/M)|^2 |\Psi(\omega_n/M)|^2,$$

where Ψ is the trigonometric polynomial $\Psi(x) = \sum_{k=M_1}^{M_2} a_k \mathrm{e}^{-2\pi \mathrm{i} k x}$ satisfying $d_1 \|\Psi\|^2 \leq \|f\|^2 \leq d_2 \|\Psi\|^2$. Thus

$$\begin{aligned}
\langle \mathcal{S} f, f \rangle &\leq \sup_{|\omega|\leq K/M} |\hat{\phi}(\omega)|^2 \sup_{|t|\leq K/M} |\Psi(t)|^2 \left(\frac{1}{M} \sum_{n=1}^{N} \frac{\omega_{n+1} - \omega_{n-1}}{2} \right) \\
&= \frac{2K}{M} \sup_{|\omega|\leq K/M} |\hat{\phi}(\omega)|^2 \sup_{|t|\leq K/M} |\Psi(t)|^2 \\
&\leq \frac{2K d_2}{M} \sup_{|t|\leq K/M} |\Psi(t)|^2,
\end{aligned}$$

where the final inequality follows from (5.1). Using the definition (3.5) of $C_1(\Omega, \mathrm{U})$, we now obtain

$$C_1(\Omega, \mathrm{T}) \leq C_1(\Omega, \mathrm{U}) \leq \frac{2K d_2}{M d_1} \inf_{\Psi \in \mathrm{V}} \left\{ \frac{\sup_{|t|\leq K/M} |\Psi(t)|^2}{\|\Psi\|^2} \right\},$$

where

$$\mathrm{V} = \left\{ \sum_{k=M_1-M_3}^{M_2-M_3} a_k \mathrm{e}^{2\pi \mathrm{i} k x} : a_k \in \mathbb{C} \right\}, \quad M_3 = \left\lceil \frac{M_1 + M_2}{2} \right\rceil.$$

Since $M_2 - M_1 \leq M$ we have $|\Psi(t)|^2 \leq (M+1)\|\Psi\|^2$, and therefore

$$C_1(\Omega, \mathrm{T}) \leq \frac{d_2}{d_1}(2K+1) \inf_{\Psi \in \mathrm{V}} \left\{ \frac{\sup_{|t|\leq K/M} |\Psi(t)|^2}{\sup_{|t|\leq 1/2} |\Psi(t)|^2} \right\}. \tag{6.1}$$

We shall return to this in a moment. First, however, let us consider $C_2(\Omega)$. We wish to show that $C_2(\Omega) \geq c$ for any Ω for some $c > 0$. Suppose first that Ω is (K, δ)-dense. Then by Lemma 4.2,

$$C_2(\Omega) \geq \left(\sqrt{1 - \|\hat{f}\|^2_{\mathbb{R}\backslash I} / \|f\|^2} - \delta \right)^2, \quad \forall f \in \mathrm{H},$$

where $I = (-K + \frac{1}{2}\delta, K - \frac{1}{2}\delta)$. Let $f(x) = \mathbb{I}_{[0,1]}(x)$, so that $\hat{f}(\omega) = \mathrm{e}^{-\mathrm{i}\pi\omega}\mathrm{sinc}(\omega\pi)$. Then we get

$$C_2(\Omega) \geq \left(\sqrt{1 - \frac{2}{\pi^2(K - \frac{1}{2})}} - \delta \right)^2 .$$

Now suppose that Ω is not (K, δ) dense. Then there exists an $n = 1, \ldots, N$ such that $\omega_{n+1} - \omega_n \geq \delta$, and therefore $\mu_n \geq \delta/2$. Hence

$$C_2(\Omega)\|f\|^2 \geq \langle \mathcal{S}f, f \rangle \geq \frac{1}{2}\delta |\hat{f}(\omega_n)|^2, \quad \forall f \in \mathrm{H}.$$

Picking $f(x) = \mathrm{e}^{2\pi\mathrm{i}\omega_n x}$, we therefore obtain $C_2(\Omega) \geq \frac{1}{2}\delta$. Hence in general

$$C_2(\Omega) \geq \max\left\{ \frac{1}{2}\delta, \left(\sqrt{1 - \frac{2}{\pi^2(K - \frac{1}{2})}} - \delta \right)^2 \right\}, \quad \forall \delta \in (0,1).$$

Since $1 - \frac{2}{\pi^2(K-\frac{1}{2})} > 0$, and since $\delta > 0$ was arbitrary, we now find that $C_2(\Omega) \geq c^2$ for any Ω. Combining this with (6.1), we now find that

$$C(\Omega, \mathrm{T}) \geq c\sqrt{\frac{d_1}{d_2 K}} \sup_{\Psi \in \mathrm{V}} \left\{ \frac{\sup_{|t| \leq 1/2} |\Psi(t)|}{\sup_{|t| \leq K/M} |\Psi(t)|} \right\}.$$

To complete the proof, we first note that

$$\min\{M_2 - M_3, M_3 - M_1\} \geq \frac{M_2 - M_1 - 1}{2}.$$

Thus, V contains all trigonometric polynomials of degree $\lfloor \frac{M_2 - M_1 - 1}{2} \rfloor \geq \frac{M_2 - M_1}{2} - 1$. An application of Lemma 6.3 now gives the result. ☐

Remark 6.4 In [4] it was shown that the reconstruction constant of GS (and therefore NUGS in the case of Fourier frames) is essentially a universal quantity. Specifically, any reconstruction algorithm that is so-called *perfect* must have a condition number that is at least that of the GS reconstruction constant. One can establish an analogous result in the (K, δ)-dense setting. Thus NUGS is essentially an optimal method for the problem. Moreover, noting Theorem 6.1, we see that to recover wavelet coefficients up to scale R stably and accurately, it is necessary to take samples from a bandwidth K that is at least 2^{R-1}, regardless of the method used.

7 Numerical examples

In this final section, we present several numerical examples to illustrate the NUGS framework. We will focus on the following three nonuniform sampling schemes:

1) Jittered sampling: Let $K > 0$and let $\eta, \epsilon \in (0,1)$ be such that $\epsilon + 2\eta < 1$. Set $\tilde{N} = \lfloor \frac{K}{\epsilon} \rfloor$ and $N = 2\tilde{N} + 1$. Jittered sampling scheme is given by

$$\Omega_N = \{\omega_1, \ldots, \omega_N\},$$

where

$$\omega_n = n\epsilon + \eta_n, \quad n = -\tilde{N}, \ldots, \tilde{N},$$

and where $\eta_n \in (-\eta, \eta)$ is chosen uniformly at random. Note that Ω_N is $(K + \eta, \delta)$-dense, where $\delta = \epsilon + 2\eta$. This sampling scheme is a standard model for jitter error in MRI caused by the measurement device not scanning exactly on a uniform grid [23].

2) Log sampling: Let $K > 0$, and let ν and δ be fixed parameters such that $2 \times 10^{-\nu} < \delta$. Set $\tilde{N} = \left\lceil -\frac{\log_{10} K+\nu}{\log_{10}(1-\delta/K)} \right\rceil$ and $N = 2(\tilde{N}+1)$. Log sampling scheme is given by

$$\Omega_N = \{-\omega_n\}_{n=0}^{\tilde{N}} \cup \{\omega_n\}_{n=0}^{\tilde{N}},$$

where

$$\omega_n = 10^{-\nu+\frac{n}{\tilde{N}}(\log_{10} K+\nu)}, \quad n = 0, \dots, \tilde{N}.$$

Note that this gives a (K, δ)-dense sampling sequence. This sampling scheme is a one-dimensional model for a two- or three-dimensional spiral sampling trajectory. Such trajectories are popular in MRI applications (see §1).

3) Seip's frame: For a given $N \in \mathbb{N}$, set

$$\Omega_N = \{\omega_n\}_{n=-1}^{-N} \cup \{\omega_n\}_{n=1}^{N},$$

where

$$\omega_n = n(1 - |n|^{-1/2}), \quad |n| \geq 1.$$

In [37], it is shown that the infinite set of frequencies $\Omega = \Omega_\infty$ gives rise to a Fourier frame with density $\delta = 1$.

The main result proved in §5 is that one requires a linear scaling of the bandwidth K or truncation index N with the parameter $M = 2^R$ for stable reconstruction in wavelet subspaces. This is illustrated in Table 1 for the Haar and DB4 wavelets. Note that the constant of the scaling is roughly 1/2, i.e. K (or N) behaves like $\beta 2^R$ with $\beta \approx 1/2$. In the case of Haar wavelets, this is due to the explicit estimates of Theorem 5.8.

T	Ω	2^R	32	64	128	256	512	1024	T	Ω	2^R	32	64	128	256	512	1024
Haar	Log	K	16	32	64	128	256	512	DB4	Log	K	16	32	64	128	256	512
	Frame	N	20	38	72	139	272	535		Frame	N	20	38	72	139	272	535

Table 1: For a given number of reconstruction vectors 2^R, the smallest value of K (or N) is shown such that the reconstruction constant $C(\Omega, \mathrm{T})$ is at most 100, where the reconstruction constant is estimated by using the results given in §4.3. This is done for different reconstruction spaces T − Haar and DB4 − and for different sampling schemes Ω: Seip's frame sequence and log sampling scheme with $\delta = 0.95$ and $\nu = 0.33$.

Theorem 6.1 also addresses such scaling by providing a lower estimate. In particular, it shows that if the scaling β is less than 1/2 then exponential instability necessarily results in the reconstruction, regardless of the wavelet basis used. This is shown in Table 2 for both Haar and DB4 wavelets. Note also that in the unstable regime, i.e. $\beta < 1/2$, the reconstruction $\tilde{f}$ is also far from quasi-optimal.

T	c_0 K	0.3125 20	0.3750 24	0.4375 28	0.5000 32	0.5625 36	0.6250 40
Haar	$\kappa(A)$	5.8569e15	2.9255e12	1.8347e05	1.7835	1.6474	1.5768
	$\frac{\Vert f-\tilde{f}\Vert}{\Vert f-\mathcal{P}_\mathrm{T} f\Vert}$	8.6294e04	7.3412e04	14.4886	1.0016	1.0016	1.0016
DB4	$\kappa(A)$	5.0079e15	2.6583e12	1.2918e05	1.6126	1.4744	1.4355
	$\frac{\Vert f-\tilde{f}\Vert}{\Vert f-\mathcal{P}_\mathrm{T} f\Vert}$	4.0459e06	3.2764e06	303.3421	1.0013	1.0009	1.0008

Table 2: The condition number $\kappa(A)$ and the error $\|f - \tilde{f}\|/\|f - \mathcal{P}_\mathrm{T} f\|$ are shown for different bandwidths $K = c_0 2^R$ and different reconstruction spaces: Haar and DB4 wavelets, where $2^R = 64$ is taken. The jittered sampling scheme is used for $\epsilon = 0.6$ and $\eta = 0.15$, and the function $f(x) = 1/2\cos(4\pi x)$ is tested.

Table 3 considers the case of Haar wavelet reconstructions more closely for the three different sampling schemes, and in particular, the magnitude of the reconstruction constant $C(\Omega, \mathrm{T})$. Recall in general that $\|f - \tilde{f}\| \leq C(\Omega, \mathrm{T})\|f - \mathcal{P}_{\mathrm{T}} f\|$, where $\tilde{f}$ is the reconstruction. The table suggests that this estimate is reasonably sharp: for the function considered it is less than four times the true value of $\|f - \tilde{f}\|/\|f - \mathcal{P}_{\mathrm{T}} f\|$ in all cases. Recall also from §3.4 that $C(\Omega, \mathrm{T})$ can be approximated by a limiting process. This is also shown in the table. Moreover, in the (K, δ)-dense case, we see that the estimate $C(\Omega, \mathrm{T}) \leq (1+\delta)/\sqrt{C_1(\Omega, \mathrm{T})}$ is also reasonably good (see the discussion in §4.3). Finally, the table also compares these estimate to the explicit bound derived in Theorem 5.8. As it is evident, this bound is also reasonably good, being that it is only roughly four times larger than the exact value of $C(\Omega, \mathrm{T})$.

Ω	K	$\lvert\Omega\rvert$	2^R	$\|f - \tilde{f}\|$	$\|f - \mathcal{P}_{\mathrm{T}} f\|$	$\frac{\|f-\tilde{f}\|}{\|f-\mathcal{P}_{\mathrm{T}} f\|}$	$\kappa(A)$	$\frac{\sigma_{\max}(A_{4096})}{\sigma_{\min}(A)}$	$\frac{1+\delta}{\sigma_{\min}(A)}$	$\frac{\pi}{2}\frac{1+\delta}{1-\delta}$
Jittered	32	108	64	6.108029e-2	6.086270e-2	1.003575	1.550640	3.722720	4.789203	14.137167
	64	215	128	3.049139e-2	3.046354e-2	1.000914	1.568731	3.840036	4.940129	
	128	428	256	1.523943e-2	1.523580e-2	1.000238	1.595984	3.914947	5.036500	
	256	855	512	7.618892e-3	7.618401e-3	1.000065	1.591625	4.157735	5.348841	
Log	32	350	64	6.107981e-2	6.086270e-2	1.003567	1.659066	3.415123	4.393487	14.137167
	64	814	128	3.049133e-2	3.046354e-2	1.000912	1.682514	3.468100	4.461641	
	128	1850	256	1.523941e-2	1.523580e-2	1.000237	1.694585	3.489929	4.489723	
	256	4146	512	7.618890e-3	7.618401e-3	1.000064	1.700702	3.504058	4.507899	
Frame	32	76	64	6.107987e-2	6.086270e-2	1.003568	2.567407	3.445520	×	×
	64	144	128	3.049194e-2	3.046354e-2	1.000932	2.520349	3.318792		
	128	278	256	1.524057e-2	1.523580e-2	1.000313	2.621085	3.588619		
	256	544	512	7.618910e-3	7.618401e-3	1.000067	2.553133	3.404633		

Table 3: The function $f(x) = \cos(6\pi x) + 1/2\sin(2\pi x)$ is reconstructed by NUGS with Haar wavelets for different sampling schemes Ω and different bandwidths K. Jittered sampling scheme is used for $\epsilon = 0.6$ and $\eta = 0.1$; and log sampling scheme is used for $\delta = 0.8$ and $\nu = 0.4$. In the last three columns, different estimates for the reconstruction constant are computed, by using the results from the Sections §3.4, 4.3 and 5.2.2.

We now wish to exhibit the advantage of NUGS: namely, it allows one to reconstruct in a subspace T that is well suited to the function to be recovered. In Figures 1 and 2 we consider the reconstruction of two functions using different wavelets. The first function is periodic, hence we use periodic wavelets, and the second is nonperiodic, and therefore we use boundary wavelets. Note that in all cases exactly the same set of measurements is used.

As is evident, increasing the wavelet smoothness leads to a smaller error. This is due to the important property of NUGS described in Corollary 5.7. Namely, since NUGS is quasi-optimal and since it requires only a linear scaling for wavelet bases, it obtains optimal approximation rates in terms of the sampling bandwidth.

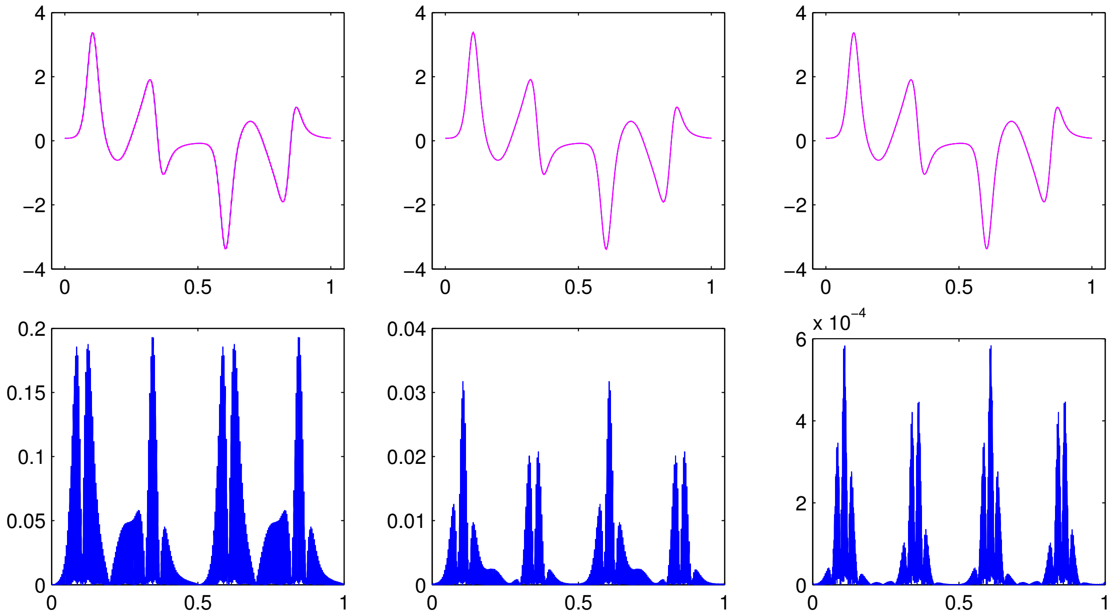

Figure 1: A smooth, periodic function reconstructed by Haar, periodic DB2 and periodic DB4 wavelets, from left to right. Above is the reconstruction $\tilde{f}$ (magenta) and the original function f (blue), and below is the error $|f - \tilde{f}|$. In all experiments, the same jittered sampling scheme is used, with $K = 128$ and $2^R = 256$.

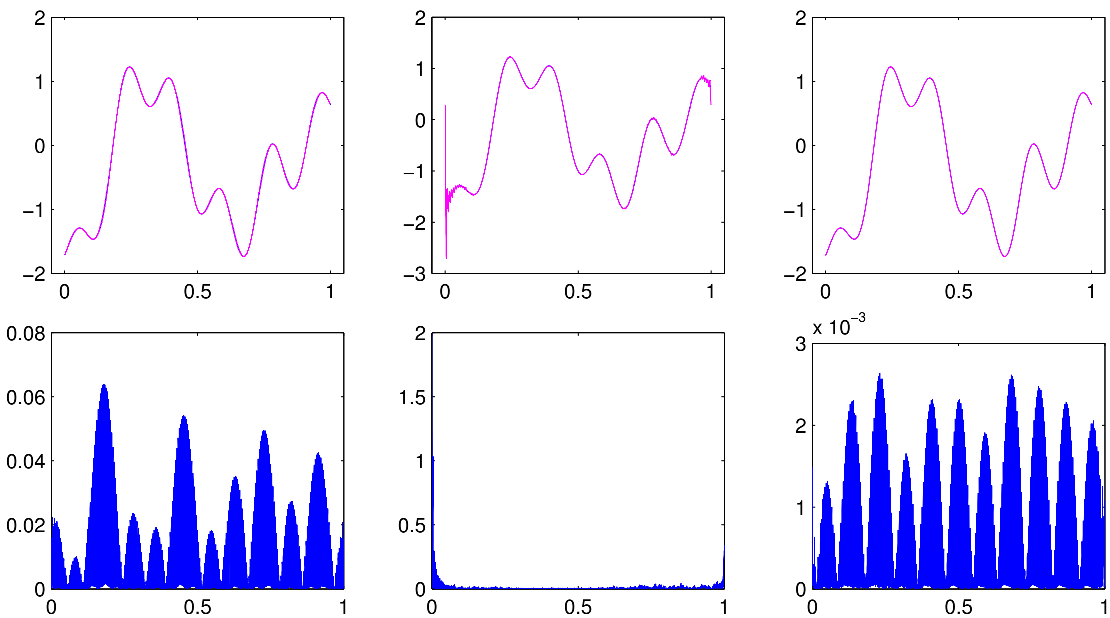

Figure 2: A smooth, nonperiodic function reconstructed by Haar, periodic DB2 and boundary DB2, from left to right. Above is the reconstruction $\tilde{f}$ (magenta) and the original function f (blue), and below is the error $|f - \tilde{f}|$. In all experiments, the same jittered sampling scheme is used, $K = 128$ and $2^R = 256$.

Next we consider the effect of noise on the NUGS reconstruction. In Table 4 we compare the actual error in reconstructing f from noisy measurements to the bound provided by the reconstruction constant $C(\Omega, \mathrm{T})$. As it is evident, the bound is reasonably close to the true noise value. We also note the robustness of NUGS with respect to noise. This is further illustrated in Figure 3, where we plot the reconstruction of a function f from noisy measurements. Even in the presence of large noise with $\eta = 0.1$, we obtain a good approximation to f.

T	η	$\|f - F(f+\eta h)\|$	estimate	T	η	$\|f - F(f+\eta h)\|$	estimate	T	η	$\|f - F(f+\eta h)\|$	estimate
Haar	0	4.4814e-2	9.4811e-2	DB2p	0	3.0899e-3	6.5489e-3	DB2b	0	4.6985e-3	9.6869-3
	0.05	6.6628e-2	2.0065e-1		0.05	4.9255e-2	1.1259e-1		0.05	6.9719e-2	1.1521e-1
	0.1	1.0830e-1	3.0650e-1		0.1	9.8086e-2	2.1867e-1		0.1	1.3918e-1	2.2073e-1
	0.2	2.0221e-1	5.1819e-1		0.2	1.9609e-1	4.3079e-1		0.2	2.7826e-1	4.3178e-1
	0.4	3.9689e-1	9.4158e-1		0.4	3.9213e-1	8.5204e-1		0.4	5.5613e-1	8.5386e-1

Table 4: The estimates $\tilde{C}(\Omega, \mathrm{T})\,(\|f - \mathcal{P}_{\mathrm{T}} f\| + \eta\|h\|)$ are computed for $f(x) = \cos(8\pi x) - 2\sin(2\pi x)$ and $h(x) = \sin(10\pi x)\mathbb{I}_{[0,1]}/\|\sin(10\pi x)\|$, where $\tilde{C}(\Omega, \mathrm{T}) = C_3(\Omega, \mathrm{T}_{4096})/C_1(\Omega, \mathrm{T}_{128})$ (see the Section §4.3), and Ω is the log sampling scheme with $K = 128$, $\delta = 0.95$, $\nu = 0.33$ and $N = 1512$. The computation is done for different reconstruction spaces $\mathrm{T} = \mathrm{T}_{128}$ with Haar, periodic DB2 and boundary DB2 functions.

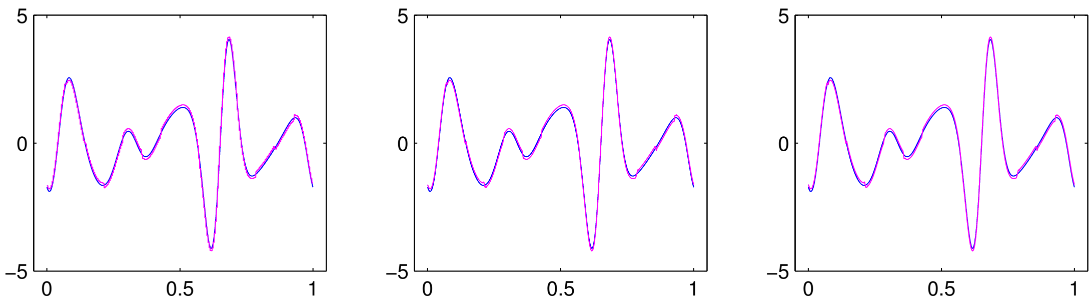

Figure 3: The function $f(x) = -\exp((\cos(6\pi x)) + \sin(4\pi x))\cos(10\pi x) + \cos(4\pi x)$ (blue) and the reconstruction $F(f + \eta h)$ (magenta), where $h(x) = \mathrm{sinc}(14\pi(x - 0.5))\mathbb{I}_{[0,1]}/\|\mathrm{sinc}(14\pi(x - 0.5))\|$ and $\eta = 0.1$. The log sampling scheme is used for $\delta = 0.95$, $\nu = 0.33$, $K = 256$ and $N = 3398$. From left to right different reconstruction basis are used for $2^R = 256$: Haar, periodic DB3 and boundary DB3.

Finally, we compare the NUGS reconstruction to the popular gridding algorithm [26, 36, 40]. Note that in the gridding algorithm, the function f is approximated on an equispaced grid by the sum

$$f(m/M) = \int_{-K}^{K} \hat{f}(\omega) \mathrm{e}^{2\pi \mathrm{i}\omega m/M} \,\mathrm{d}\omega \approx \sum_{n=1}^{N} \hat{f}(\omega_n) \mathrm{e}^{2\pi \mathrm{i}\omega_n m/M} \mu_n, \quad m = 0, \ldots, M,$$

which can be evaluated efficiently using NUFFTs. A global representation of f on $[0, 1]$ can then be computed via a standard inverse FFT. Unfortunately, this reconstruction is plagued by artefact, even when the original function is periodic. This is shown in the left panels of Figures 4 and 5. Alternatively, one can use the NUGS reconstruction with wavelets. As shown in these figures, this gives a far superior reconstruction of f, even in the case of discontinuous functions with sharp peaks (see Figure 5). Recall also that the NUGS reconstruction, much like the gridding approximation, can also be computed efficiently using NUFFTs (see Remark 3.8). Hence, using the same measurement data, and with roughly the same computational cost, we are able obtain a vastly superior reconstruction.

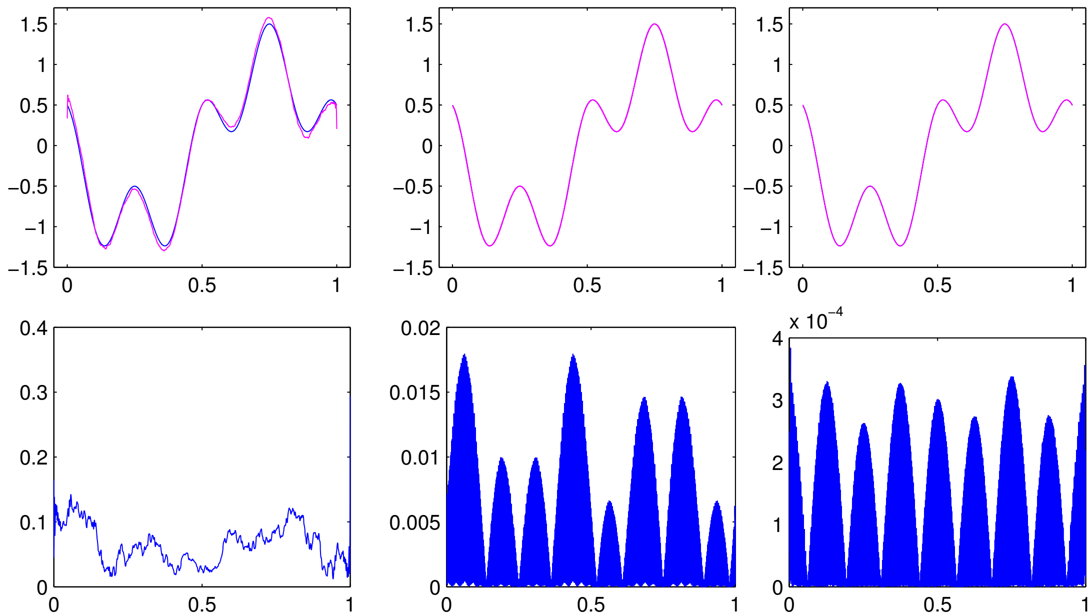

Figure 4: A periodic function $f(x) = 1/2\cos(8\pi x) - \sin(2\pi x)$ is reconstructed by gridding (left) and NUGS with Haar (middle) and DB2 (right) wavelets for $2^R = 512$. The lower pictures show the error $|f - \tilde{f}|$. The jittered sampling scheme is used for $\epsilon = 0.7$, $\eta = 0.14$ and $K = 256$.

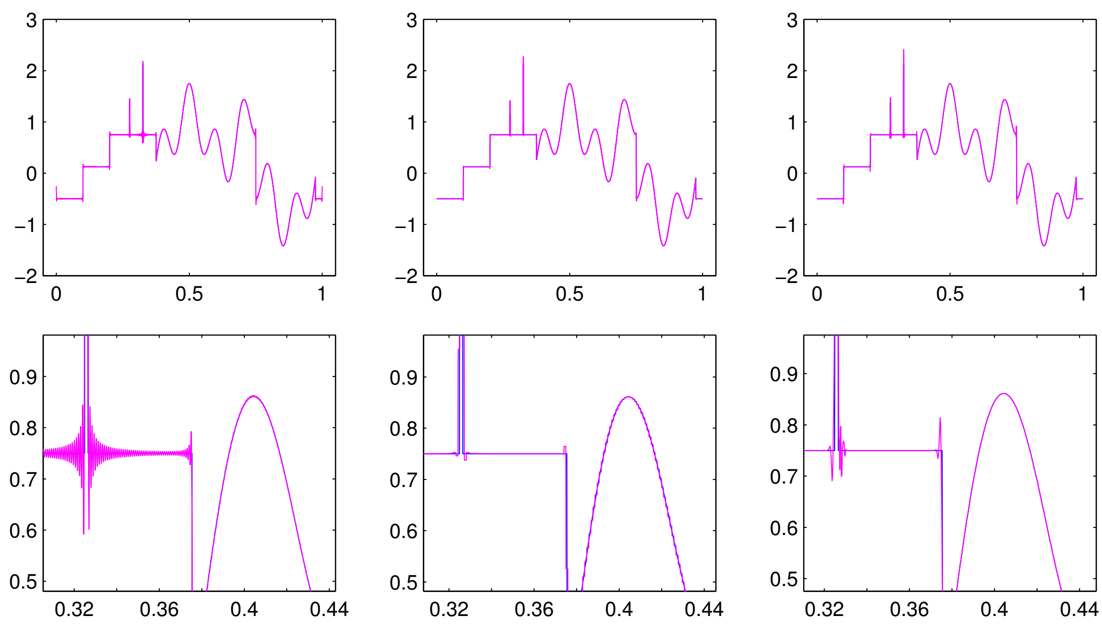

Figure 5: A discontinuous function reconstructed by gridding, and NUGS with Haar and DB4 wavelets (from left to right). The reconstruction is in magenta and original in blue. Below, a close-up is shown. The jittered sampling is used for $\epsilon = 0.75$ and $\eta = 0.1$ where $K = 1024$ and NUGS is used for $2^R = K$.

8 Conclusions and future work

The purpose of this paper was to introduce a framework, NUGS, for stable reconstructions in arbitrary finite-dimensional subspaces T from nonuniform Fourier samples. We have shown that this is always possible provided the samples are (K, δ)-dense or arise from a Fourier frame, and provided the bandwidth K or index N is taken sufficiently large in relation to T. Moreover, for the important case where T consists of wavelets, we have shown that a linear scaling of K or N with the dimension $M = 2^R$ suffices, but that this scaling cannot be below a certain critical threshold, otherwise exponential instability necessarily occurs.

There are several topics for future work. First, much of the theory developed in this paper extends to higher dimensions. This is an important topic, since the primary motivation for this work deals with the recovery of two- and three-dimensional images. It will be presented elsewhere. Note that in higher dimensions, it is also important to analyze other reconstruction spaces besides wavelets, such as curvelets and shearlets.

Second, there is the important question of how the reconstruction constant $C(\Omega, \mathrm{T})$ behaves for other common choices of subspace T. In [3] it was shown when T consists of polynomials of degree at most M, then one requires $\mathcal{O}\left(M^2\right)$ uniform Fourier samples to ensure boundedness of $C(\Omega, \mathrm{T})$. This quadratic scaling is in fact necessary, as was shown in [8]. Similarly, when T consists of trigonometric polynomials, it was shown in [4] that a linear scaling suffices whenever samples arise from a Fourier frame. We believe both results can be extended to the (K, δ)-dense case, and leave this for future work.

A third topic for future work involves the choice of the operator $\mathcal{S}$. The theory developed in §3 allows for many other choices of $\mathcal{S}$ than that was considered in the latter half of the paper (namely (3.8) with weights given by (4.1)). It is possible that different choices, possibly depending on the subspace T, may yield improvements in the magnitude of the reconstruction constant.

Finally, as discussed in §1, the eventual aim of this work is to combine the theory developed here with compressed sensing tools to allow for recovery of compressible images from relatively few nonuniform Fourier samples. This is also work in progress. For an extensive discussion in the case of uniform Fourier measurements we refer to [6].

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