5 The Wave Equation

Waves are extremely common in the physical world. Examples include surface disturbance of a body of fluid, vibration of string instruments and pressure perturbations in air that convey sound. In all these cases, if the amplitude of the disturbance is sufficiently small, the perturbation variable $\phi(\mathbf{x}, t)$ characterising the disturbance will satisfy the *wave equation*:

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi \tag{5.1}$$

where c is the (phase) speed of propagation of maxima and minima of a sinusoidal wave form. In these examples, extra non-linear terms will need to be introduced if the disturbance becomes large, and the wave equation is only a kind of lowest order approximation. This is entirely analogous to the behaviour of a point particle in mechanics, that is trapped in a local minimum of some potential function and performing *small* motions: regardless of the overall form of the potential, to lowest order approximation (e.g. in Taylor series expansion) any (suitably differentiable) function appears quadratic around a local minimum, and the particle will thus execute *simple harmonic motion* if the amplitude is small. But the wave equation also has genuinely fundamental significance in other areas: for example in electromagnetic theory, Maxwell's equations imply the the electromagnetic potentials must satisfy the wave equation in regions free of sources, and this lead to the understanding of (classical) light as an electromagnetic phenomenon – a truly awesome discovery at the time.

5.1 Vibrations of a uniform string

To bring the discussion down to earth, let's think of the example of a violin string of length L. So we take the spatial region Ω to just be the interval $[0, L]$ with $x \in [0, L]$ denoting the location along the string.

We'll start by illustrating the physical origin of the wave equation in this example. Consider a small transverse oscillation of our string with ends fixed at $x = 0$ and $x = L$. To keep things simple, let's assume that the string is uniform with constant mass per unit length μ and is perfectly elastic. That's pretty much true of a well–made violin string. We'll also assume that the string only performs small transverse oscillations $\phi(x, t)$, so that we only need to work to first order in ϕ. Consider a small element δs of string between x (point A) and $x + \delta x$ (point B) having mass $\mu\delta x$. Let θ_A, θ_B be the angles at the ends and let T_A, T_B be the outward pointing tangential tension forces acting on δs. Since the motion is transverse, the total force along the string is zero so

$$T_A \cos\theta_A = T_B \cos\theta_B = T = \text{constant}\,. \tag{5.2}$$

In the transverse direction, Newton's second law gives

$$\mu\delta x \frac{\partial^2 \phi}{\partial t^2} = T_B \sin\theta_B - T_A \sin\theta_A \tag{5.3}$$

Dividing through by the quantities in equation (5.2) gives

$$\frac{\mu\delta x}{T}\frac{\partial^2 \phi}{\partial t^2} = \frac{T_B \sin\theta_B}{T_B \cos\theta_B} - \frac{T_A \sin\theta_A}{T_A \cos\theta_A} = \tan\theta_A - \tan\theta_B\,. \tag{5.4}$$

Now

$$\tan\theta_B - \tan\theta_A = \left.\frac{\partial\phi}{\partial x}\right|_B - \left.\frac{\partial\phi}{\partial x}\right|_A \approx \frac{\partial^2\phi}{\partial x^2}\,\delta x\,,$$

so that after dividing through by $\mu\,\delta x/T$, equation (5.4) becomes

$$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2\phi}{\partial x^2}\,, \qquad \text{where } c^2 = T/\mu. \tag{5.5}$$

Thus we have derived the wave equation in 1+1 dimensions. The constant c has units of a velocity and is called the *phase speed*. Note that from the role of Newton's law in the above derivation, for a unique solution (in addition to the BCs of fixed endpoints $\phi(0,t) = \phi(L,t) = 0$ for all t) we would expect to have to provide both the initial position $\phi(x,0)$ and the initial velocity $\partial\phi/\partial t(x,0)$ for $0 < x < L$ of all points along the string. We'll say more about uniqueness in section 5.2.

We now wish to solve the 1+1 dimensional wave equation subject to the boundary conditions

$$\phi(0,t) = \phi(L,t) = 0 \tag{5.6}$$

and initial conditions

$$\phi(x,0) = f(x)\,, \qquad \frac{\partial\phi}{\partial t}(x,0) = g(x) \tag{5.7}$$

for some given functions f and g, representing the string's initial shape and velocity. Separation of variables $\phi(x,t) = X(x)T(t)$ leads to the o.d.e.s

$$X'' = -\lambda X \qquad T'' = -c^2\lambda T \tag{5.8}$$

in terms of a separation constant λ. These are solved by sines and cosines, and the boundary conditions $\phi(0,t) = \phi(L,t) = 0$ enforce require that

$$\lambda = \frac{n^2\pi^2}{L^2} \qquad \text{for some } n \in \mathbb{N}. \tag{5.9}$$

Consequently our individual solution is

$$\phi_n(x,t) = \sin\frac{n\pi x}{L}\left[A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right] \tag{5.10}$$

which, for a fixed value of n is known as a *normal mode* of the oscillation. In particular, the lowest non–trivial value $n = 1$ is known as the *fundamental mode*. We see that all the frequencies of oscillation $\omega_n = n\pi c/L$ are integer multiples of the fundamental frequency ω_1.

Summing over possible separation constants, the general solution is a sum over the normal modes

$$\phi(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left[A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right]. \tag{5.11}$$

As usual, the constants A_n and B_n are fixed by the inhomogeneous initial conditions to be

$$A_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\, f(x)\, \mathrm{d}x\ , \qquad\qquad B_n = \frac{2}{n\pi c}\int_0^L \sin\frac{n\pi x}{L}\, g(x)\, \mathrm{d}x\,. \tag{5.12}$$

For example, if we pluck the string, pulling it back to height h in the middle and releasing it from rest, then we have the initial conditions

$$f(x) = \begin{cases} 2hx/L & 0 \leq x \leq L/2 \\ 2h(L-x)/L & L/2 \leq x/leqL \end{cases}$$

while $g(x) = 0$. We computed the Fourier coefficients of this plucked string function $f(x)$ in the problem sets, finding

$$\hat{f}_n = \begin{cases} (-1)^{(n+1)/2}\dfrac{8h}{n^2\pi^2} & \text{when } n \text{ is odd} \\ 0 & \text{else.} \end{cases} \tag{5.13}$$

Using this result we have

$$\phi(x,t) = \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\,\sin\left[\frac{(2n-1)\pi x}{L}\right]\,\sin\frac{n\pi ct}{L}\,. \tag{5.14}$$

It's a good idea to check that this does indeed satisfy the wave equation, boundary conditions and initial conditions.

5.2 Energetics and uniqueness

It will be helpful to derive an expression for the energy contained in the string's motion. Since the string has mass per unit length μ, its total kinetic energy at time t is

$$K(t) = \frac{1}{2}\int_0^L \mu\left(\frac{\partial\phi}{\partial t}\right)^2 \mathrm{d}x\,. \tag{5.15}$$

The string is under tension, so it will also have some potential energy whenever its profile is non–constant. Considering a small element δs of the string we have

$$T\times(\text{extension}) = T(\delta s - \delta x) = T\left[\sqrt{1+\left(\frac{\partial\phi}{\partial x}\right)^2}\ -\ 1\right]\,\delta x \tag{5.16}$$

and integrating this along the length of the string gives a potential energy contribution at time t

$$V(t) = T\int_0^L\left[\sqrt{1+\left(\frac{\partial\phi}{\partial x}\right)^2}\ -\ 1\right]\,\mathrm{d}x \approx \frac{T}{2}\int_0^L\left(\frac{\partial\phi}{\partial x}\right)^2\mathrm{d}x\,. \tag{5.17}$$

In the final approximation we used the fact that the oscillations are small. Using the fact that $c^2 = T/\mu$ we see that the total energy of the string is

$$E(t) = \frac{\mu}{2}\int_0^L\left[\left(\frac{\partial\phi}{\partial t}\right)^2 + c^2\left(\frac{\partial\phi}{\partial x}\right)^2\right]\,\mathrm{d}x\,. \tag{5.18}$$

It's a useful exercise to evaluate this energy function for the explicit solution (5.11). You should find that the kinetic and potential energies are

$$
\begin{aligned}
K(t) &= \frac{\mu\pi^2c^2}{4L}\sum_{n=1}^{\infty} n^2 \left[A_n \sin\left(\frac{n\pi ct}{L}\right) - B_n \cos\left(\frac{n\pi ct}{L}\right)\right]^2 \\
V(t) &= \frac{\mu\pi^2c^2}{4L}\sum_{n=1}^{\infty} n^2 \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right]^2
\end{aligned}
\tag{5.19}
$$

so that the total energy is

$$
E(t) = \frac{\mu^2c^2\pi^2}{4L}\sum_{n=1}^{\infty} n^2\,(A_n^2 + B_n^2)\,. \tag{5.20}
$$

Notice in particular that this total energy is independent of time; just like in simple harmonic motion, PE and KE are continuously inter–converted during the motion so that the total energy is conserved. Also notice that, in accordance with our intuition, given two modes with equal amplitudes the higher mode has the higher energy. Finally, recall that the period of oscillation (*i.e.* the period of the fundamental mode) is

$$
T = \frac{2\pi}{\omega} = 2\pi\frac{L}{\pi c} = \frac{2L}{c}. \tag{5.21}
$$

and we can average over a period to get

$$
\overline{K} = \frac{c}{2L}\int_0^{\frac{2L}{c}} K(t)\,\mathrm{d}t = \overline{V} = \frac{c}{2L}\int_0^{\frac{2L}{c}} V(t)\mathrm{d}t = \frac{E}{2}\,, \tag{5.22}
$$

so there is an *equipartition of energy* between average potential and kinetic energies.

The energy provides a good way to prove uniqueness of solutions to the wave equation in general, provided they are subject to appropriate boundary and initial conditions. To see this, let $M \cong \Omega\times[0,\infty)$ and let $\phi : M \to \mathbb{R}$ be a solution of the wave equation[24] in the interior of M, that obeys the conditions

$$
\begin{aligned}
&\phi|_{\Omega\times\{0\}} = f(x) && \text{(initial condition on } \phi \text{ itself)} \\
&\partial_t\phi|_{\Omega\times\{0\}} = g(x) && \text{(initial condition on time derivative of } \phi) \\
&\phi|_{\partial\Omega\times(0,\infty)} = h(x) && \text{(Dirichlet boundary condition at } \partial\Omega)
\end{aligned}
\tag{5.23}
$$

Notice that, as above, we have two initial conditions: at time $t = 0$ we prescribe the values both of ϕ itself and its first time derivative everywhere over space Ω. We also impose Dirichlet boundary conditions at the boundary of our compact region Ω that hold for all times.

Absorbing a factor of μ into the scaling of ϕ, we define the *energy* $E_\phi(t)$ of this wave at time $t \in (0,\infty)$ to be the integral

$$
E_\phi(t) \equiv \frac{1}{2}\int_\Omega \left(\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial t} + c^2\nabla\phi\cdot\nabla\phi\right)\mathrm{d}V \tag{5.24}
$$

[24]For simplicity, I'll only consider real–valued functions here. What I say generalizes to complex–valued waves easily – have a go!

over the spatial region Ω. This is a natural generalization of the energy of our violin string, involving a kinetic term involving the rate $\partial_t\phi$ at which each point in Ω is oscillating, and a potential term involving the tension due to spatial gradients in the wave. Differentiating under the integral and using the fact that partial derivatives commute one finds

$$\begin{aligned}\frac{dE_\phi}{dt} &= \int_\Omega \left[\frac{\partial\phi}{\partial t}\frac{\partial^2\phi}{\partial t^2} + c^2\nabla\left(\frac{\partial\phi}{\partial t}\right)\cdot\nabla\phi\right] \mathrm{d}V \\ &= \int_\Omega \frac{\partial\phi}{\partial t}\left[\frac{\partial^2\phi}{\partial t^2} - c^2\nabla^2\phi\right] \mathrm{d}V + c^2\int_{\partial\Omega}\frac{\partial\phi}{\partial t}(\mathbf{n}\cdot\nabla\phi)\,\mathrm{d}S \\ &= c^2\int_{\partial\Omega}\frac{\partial\phi}{\partial t}\,\mathbf{n}\cdot\nabla\phi\,\mathrm{d}S\end{aligned} \tag{5.25}$$

where in going to the second line we integrate by parts in the spatial variables, and in going to the last line we used the fact that ϕ solves the wave equation. Thus, if either $\mathbf{n}\cdot\nabla\phi|_{\partial\Omega} = 0$ or $\partial_t\phi|_{\partial\Omega} = 0$ so that no energy is flowing out of the region Ω, then evolution via the wave equation preserves $E_\phi(t)$.

Now we're ready for our uniqueness theorem. Suppose ϕ_1 and ϕ_2 are two solutions of the wave equation inside M that each obey the boundary conditions (5.23). Then $\psi \equiv \phi_1 - \phi_2$ solves the wave equation subject to

$$\psi|_{\Omega\times\{0\}} = \partial_t\psi|_{\Omega\times\{0\}} = \psi|_{\partial\Omega\times(0,\infty)} = 0\,. \tag{5.26}$$

In particular, the fact that $\psi|_{\partial\Omega} = 0$ for all times means that $\partial_t\psi|_{\partial\Omega} = 0$ so that $dE_\psi/dt = 0$ and therefore that

$$E_\psi(t) = \frac{1}{2}\int_\Omega\left(\frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial t} + c^2\nabla\psi\cdot\nabla\psi\right)\mathrm{d}V$$

remains constant. But since both $\partial_t\psi$ and ψ itself vanish throughout Ω at $t = 0$, evaluating this integral at the initial time gives

$$E_\psi(t) = E_\psi(0) = 0\,. \tag{5.27}$$

Finally, since $E(t)$ is the integral of a sum of non–negative quantities, the only way for $E(t)$ to vanish is if $\partial_t\psi$ and $\nabla\psi$ each vanish separately throughout Ω at all times. Thus ψ is constant on $\Omega\times[0,\infty)$ and since the initial value of ψ is zero, ψ is everywhere zero. Hence our two solutions ϕ_1 and ϕ_2 are in fact the same.

This result is useful: it says that if we manage to find a solution satisfying boundary and initial conditions as in (5.23) by any means (*e.g.* separation of variables), then we've found the only solution and we're done. As usual, if we replace the Dirichlet condition on the boundary of the spatial region Ω by a Neumann condition $\mathbf{n}\cdot\nabla\phi|_{\partial\Omega} = h(x)$ the uniqueness argument goes through unchanged up to the last step, where we conclude that our two solutions can differ at most by a constant.

5.3 Vibrations of a circular membrane

For our next example, we'll consider the vibrations of a circular drum. Let's take Ω to be the unit disc $\{(r,\theta)\in\mathbb{R}^2 \,:\, r\le 1\}$ and write the wave equation in cylindrical coordinates as

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \tag{5.28}$$

We'll suppose the drum's membrane it held fast at the boundary, so $\phi|_{r=1} = 0$ for all times t. Writing $\phi(r,\theta,t) = R(r)\,\Theta(\theta)\,T(t)$ we find

$$T'' = -c^2\lambda T\ , \qquad \Theta'' = -\mu\,\Theta\ , \qquad r(rR')' + (r^2\lambda - \mu)R = 0 \tag{5.29}$$

The T and Θ equations have the usual sinusoidal and cosinusoidal solutions, and to ensure our solution is single–valued as $\theta \to \theta + 2\pi$ we must take $\mu = m^2$ for some $m \in \mathbb{N}$. The radial equation becomes $r(rR')' + (r^2\lambda - m^2)R = 0$, which is Bessel's equation of order m. As in section 3.4.1, the solutions are

$$R(r) = a_m J_m\left(\sqrt{\lambda}r\right) + b_m Y_m\left(\sqrt{\lambda}r\right) \tag{5.30}$$

where we take $b_m = 0$ to ensure regularity at the origin. To satisfy the boundary condition at $r = 1$, we must choose the separation constant λ to be one of the

$$\lambda = k_{mi}^2 \qquad \text{where } J_m(k_{mi}) = 0 \tag{5.31}$$

so that k_{mi} is the i^{th} root of the m^{th} Bessel function $J_m(r)$.

Combining the pieces, we have the general solution

$$\begin{aligned}\phi(r,\theta,t) &= \sum_{i=1}^{\infty}\left[A_{0i}\sin(k_{0i}ct) + C_{0i}\cos(k_{0i}ct)\right] J_0(k_{0i}r)\\ &\quad+ \sum_{m=1}^{\infty}\sum_{i=1}^{\infty}\left[A_{mi}\cos m\theta + B_{mi}\sin m\theta\right]\,\sin(k_{mi}ct)\,J_m(k_{mi}r)\\ &\quad+ \sum_{m=1}^{\infty}\sum_{i=1}^{\infty}\left[C_{mi}\cos m\theta + D_{mi}\sin m\theta\right]\,\cos(k_{mi}ct)\,J_m(k_{mi}r)\end{aligned} \tag{5.32}$$

which is admittedly a bit of a mouthful. Pictures of some of the normal modes of oscillation can be seen in figure 9. If the drum's surface has initial profile and velocity

$$\phi(r,\theta,0) = f(r,\theta) \qquad \partial_t\phi(r,\theta,0) = g(r,\theta) \tag{5.33}$$

then the constants $\{A_{mi}, B_{mi}, C_{mi}, D_{mi}\}$ may be fixed by expanding both f and g in Fourier series in θ and Bessel functions in r. We recall that the Bessel functions of order m obey a Sturm–Liouville differential equation and are thus orthogonal for different values of k_{mi}. Explicitly,

$$\int_0^1 J_m(k_{mi}r)\,J_m(k_{mj})\,r\,\mathrm{d}r = \frac{1}{2}\delta_{ij}\,\left[J_{m+1}(k_{mi})\right]^2 \tag{5.34}$$

as you showed in one of the problems.

As an example, suppose that the drum is initially quiet with $\phi = 0$ but at $t = 0$ is suddenly struck in the centre, so that

$$\phi(r,\theta,0) = 0\,, \qquad \partial_t\phi(r,\theta,0) = g(r)\,, \tag{5.35}$$

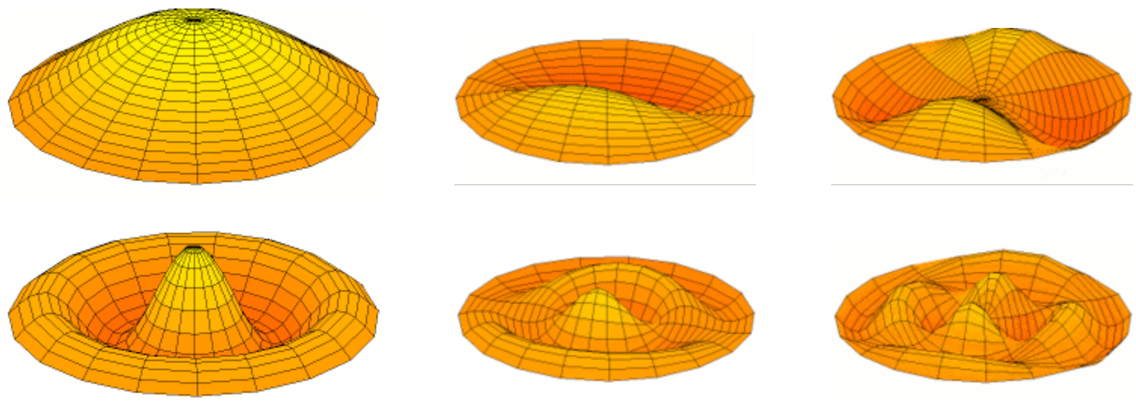

Figure 9. *Various normal modes of oscillation of a circular drum. In the left column the modes* $(m,i) = (0,1)$ *and* $(0,3)$ *are shown, the middle column shows the modes* $(m,i) = (1,1)$ *and* $(1,3)$ *and the right column shows the modes* $(m,i) = (2,1)$ *and* $(2,3)$*. These figures are taken from this Wikipedia page, where you can find animated versions.*

where the initial velocity is a function of r only. The solution is then also independent of the angle θ and the only non–vanishing constants are A_{0i}. (The C_{0i} must vanish since we need $\phi|_{t=0} = 0$. The general solution (5.32) reduces to

$$\phi(r,\theta,t) = \sum_{i=1}^{\infty} A_{0i} \sin(k_{0i}ct)\, J_0(k_{0i}r) \tag{5.36}$$

where the remaining constants A_{0i} are given by

$$A_{0i} = \frac{2}{ck_{0i}} \frac{1}{[J_1(k_{0i})]^2} \int_0^1 J_0(k_{0i}r)\, g(r)\, r\, \mathrm{d}r\,. \tag{5.37}$$

Interestingly, the fundamental frequency for a drum of general radius a is $k_{01}c/a \sim 4.8c/a$, which is higher than the fundamental frequency $\pi c/a$ of a string of length a. Also, the fundamental response of the drum is just a Bessel function, so our ears experience these functions rather frequently even if they seem unfamiliar to our brains.

5.4 Can one hear the shape of a drum?

I'm sure you can spot my non–examinable sections by now, but just in case: this is one.

We've seen that the normal modes of a violin string oscillate at frequencies that are integer multiples of the fundamental frequency $\pi c/L$ and it's easy to see that the fundamental frequencies of a rectangular membrane with sides of lengths L_1 and L_2 will be $c\pi\sqrt{m^2/L_1^2 + n^2/L_2^2}$ for $m, n \in \mathbb{N}$ if the membrane is held fast along its four edges. (If this isn't clear to you – *work it out!*) On the other hand, for a circular membrane of radius a the frequencies of the normal modes are $k_{mi}c/a$ where k_{mi} are the irregularly spaced roots of the m^{th} Bessel functions. Given a membrane of an arbitrary shape, fixed in place around its boundary, it can be a difficult problem to determine exactly what the normal modes are, particularly in higher dimensions.

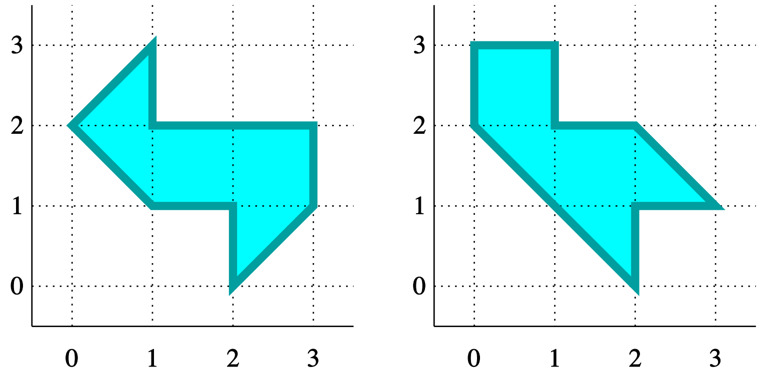

Figure 10. *These two shapes, constructed by Gordon, Webb & Wolpert following a method of Sunada, share all their eigenvalues of the Laplacian, so perfect drums made in these shapes would sound identical. Both polygons have the same area and same perimeter.*

In 1966, Mark Kac turned the question around. Instead of asking "Given a membrae, can we find its frequencies of oscillation?" he asked instead "Suppose we are given the complete set $\{\omega_I\}$ of frequencies of the normal modes of oscillation of some membrane, where I takes values in some indexing set. Can we use these to work out the domain $\Omega \subset \mathbb{R}^n$ spanned by the membrane?". More poetically, the question can be phrased "Can one hear the shape of a drum?".

As put, the answer to this question is "No". In other words, we now know that there do exist two different shapes all of whose eigenvalues of the Laplacian coincide. Such shapes are said to be *isospectral*. The first example to be found was, remarkably enough, in sixteen (!) dimensions and these two 16-dimensional isospectral shapes each turn out to play an important rôle in modern string theory: They're each responsible for one of the two weakly–coupled heterotic string theories, one of which was found by Prof. Michael Green here in Cambridge, together with Prof. John Schwarz in Caltech. The fact that they are isospectral is in fact important for consistency of the two theories. Later, more and more examples of isospectral shapes were found; the simplest known pair in two spatial dimensions is shown in figure 10.

Not being put off by this negative result, people then asked whether the question could be answered 'yes' under special conditions. For example, it's known that if the membrane is *convex*[25] and has boundary specified by a real–analytic function, then one can hear its shape. More generally, you might wonder exactly how much information about Ω can be retrieved from knowing all the frequencies ω_I. For example, Weyl showed that the total area $A(\Omega)$ of the surface of a drum is given by

$$A(\Omega) = 4\pi^2 \lim_{\lambda_0\to\infty} \frac{N(\lambda_0)}{\lambda_0}\,, \tag{5.38}$$

where $N(\lambda_0)$ is the number of eigenfrequencies less than the scale λ_0. Thus the area is related to the asymptotic growth of the number of eigenvalues. Weyl also generalized this

[25] A region $\Omega \subset \mathbb{R}^n$ is convex if, given any pair of points $x_1, x_2 \in \Omega$ the straight line segment joining x_1 to x_2 is also entirely contained in Ω.

formula to $n+1$ dimensions, and conjectured that the subleading terms in this limit were related to the perimeter of the drum (or the volume of $\partial\Omega$ in higher dimensions).

The whole field of studying the geometry of some Ω by studying its eigenfrequencies is known as *spectral geometry.* You can take a course on it in Part III.

5.5 Wave reflection and transmission

If the medium through which the wave is propagating has spatially varying properties, then the properties of the wave will change too, with for example the possibility of partial reflection at an interface.

Suppose we have an (infinite) string with density $\mu = \mu_-$ for $x < 0$ and $\mu = \mu_+$ for $x > 0$ and consider small transverse deflections. Resolving forces horizontally as before, we see that the tension τ must remain constant (even with density variations) and so the wave speed $c_\pm \equiv \sqrt{\tau/\mu_\pm}$ differs on either side of $x = 0$. Consider an incident wave propagating to the right from $x = -\infty$. The most general form is

$$\phi_I(x,t) = A_I \cos[\omega(t - x/c_-) + \xi_I] \tag{5.39}$$

with frequency ω, amplitude A_I and phase shift ξ_I; the subscript I denotes that this is the "incident" wave. It is convenient to represent such waves in terms of a complex exponential

$$\phi_I(x,t) = \mathrm{Re}\left(I \exp\left[\mathrm{i}\omega\left(t - x/c_-\right)\right]\right) , \tag{5.40}$$

where Re denotes the real part. Here, we've introduced the complex number I whose modulus is the amplitude A_I and whose phase is the phase shift ξ_I of (5.39). Again, the use of the capital letter I reminds us that this is the incident wave; we'll soon meet complex numbers R and T denoting the "reflected" and "transmitted" waves. If necessary, we'll use lowercase r and i to denote the real and imaginary parts of these quantities, so that $I = I_r + \mathrm{i}I_i$ for example.

On arrival at $x = 0$ some of the incident wave will be *transmitted* and so continue to propagate to the right into $x > 0$, while some will be *reflected* and so propagate back to the left. Both of these waves may have different amplitudes and phases than those of the incident wave. However, they must have the same frequencies if the string is to stay together at all times (in particular at the point $x = 0$). Using subscripts T for "transmitted" and R for "reflected" we write

$$\begin{aligned} \phi_T(x,t) &= \mathrm{Re}\left(T \exp\left[\mathrm{i}\omega\left(t - \frac{x}{c_+}\right)\right]\right) \\ \phi_R(x,t) &= \mathrm{Re}\left(R \exp\left[\mathrm{i}\omega\left(t + \frac{x}{c_-}\right)\right]\right) \end{aligned} \tag{5.41}$$

The complex coefficients T and R define the new amplitudes and phases via their moduli and arguments. These coefficients are determined by the following physical matching conditions at $x = 0$:

– We assume the string does not break, so the displacement at $x = 0$ must be continuous *for all time*. That is,

$$\phi_I|_{x=0^-} + \phi_R|_{x=0^-} = \phi_T|_{x=0^+} \, .$$

Using the fact that if $\mathrm{Re}\left(A e^{\mathrm{i}\omega t}\right) = \mathrm{Re}\left(B e^{\mathrm{i}\omega t}\right)$ for all t then $A = B$ as complex numbers, we get

$$I + R = T \, . \tag{5.42}$$

– The point $x = 0$ has no inertia (compare with a different situation in probs. 2!) and thus the total vertical force at $x = 0$ vanishes. Hence

$$\tau \left.\frac{\partial \phi}{\partial x}\right|_{x=0^-} = \tau \left.\frac{\partial \phi}{\partial x}\right|_{x=0^+} ,$$

or in other words

$$\frac{R}{c_-} - \frac{I}{c_-} = -\frac{T}{c_+} \, . \tag{5.43}$$

Equations (5.42) & (5.43) suffice to fix the two complex numbers R and T in terms of I. Solving we get

$$R = \left(\frac{c_+ - c_-}{c_+ + c_-}\right) I \qquad \text{and} \qquad T = \left(\frac{2c_+}{c_+ + c_-}\right) I \, . \tag{5.44}$$

This solution has several interesting properties. Firstly, we see that $R_i/R_r = T_i/T_r = I_i/I_r$ so there is a simple relationship between the phases of the waves. Secondly, if $\mu_+ = \mu_-$ so that $c_+ = c_-$, we find that $R = 0$ and $T = I$ as expected: in the absence of any change in μ, the wave travels on unhindered. On the other hand, if the string to the right is very much heavier so that $\mu_+ \gg \mu_-$, then $c_+ \ll c_-$ and we find $T \sim 0$. As expected, almost all the wave is reflected. Note however that the reflected wave has phase shift π compared to the incident wave, since $R \approx -I$. Finally, if the string to the right $x > 0$ is very much lighter $\mu_+ \ll \mu_-$, then $c_+ \gg c_-$ and we find $T \sim 2I$ and $R \sim I$. In this case there is no phase shift, and we get a large amplitude of disturbance to the right. However most of the energy is still reflected (as the mass is very low on the right) so in both the asymmetrical limiting cases, most of the energy is reflected.