Part Ib Variational Principles: *A very short version*[1]

1 Extremising Functions (Hessian and Lagrange Multipliers)

For a suitably differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, the point $\mathbf{a} \in \mathbb{R}^n$ is **stationary** if $\nabla f(\mathbf{a}) = \mathbf{0}$. The **Hessian** matrix is given by $H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. The Hessian evaluated at $\mathbf{x} = \mathbf{a}$ can be used to determine the nature of the stationary point:

- all eigenvalues positive: **(local) minimum**
- all eigenvalues negative: **(local) maximum**
- some positive, some negative: **saddle**
- some eigenvalue zero: may need higher derivatives

In $\mathbb{R}^2$, can determine this purely from the signs of the determinant and trace.

Use **Lagrange multipliers** to extremise functions subject to constraints, e.g. extremise $f(x, y)$ subject to $g(x, y) = 0$, then consider $h(x, y, \lambda) = f(x, y) - \lambda\, g(x, y)$ and extremise h with respect to x, y and λ. Can generalise this for multiple constraints: use more multiplers λ_i. Can also use Lagrange multipliers with integrals (and then use Euler-Lagrange on the combined function).

2 Extremising Integrals (Euler-Lagrange)

Seek to extremise a functional $F[y] = \int_a^b f(x, y, y')dx$, where $y(a)$ and $y(b)$ are given. Perturb $y(x) \to y(x) + \epsilon\eta(x)$ with $\eta(a) = \eta(b) = 0$. Consider expansion in ϵ and write $F[y + \epsilon\eta] = f[y] + \epsilon\, \delta F + \epsilon^2\, \delta^2 F + \mathcal{O}(\epsilon^3)$. Start with the first variation δF, integrating the η' term by parts to reach the form $\int [\ldots]\eta\, dx$:

$$\delta F = \int_a^b \frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta' dx = \int_a^b \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta\, dx$$

For y to make F stationary, δF must be zero for all possible $\eta(x)$, hence the square bracket must be zero, yielding the **Euler-Lagrange equation**:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$

This can be extended in various ways for more general f:

- If more y_i: there is an E-L equation for each i: $\frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) - \frac{\partial f}{\partial y_i} = 0$
- If more x_i: then first term becomes a sum ($\sum_i \frac{\partial}{\partial x_i}\frac{\partial f}{\partial y_{x_i}}$)
- If more derivates (e.g. $f(x, y, y', y'')$, then additional terms: $\ldots - \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) + \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$

There are two first integrals of Euler-Lagrange for special f:

- If $\frac{\partial f}{\partial y} = 0$: $\frac{\partial f}{\partial y'} = \mathrm{const}$ (this one is immediate)
- If $\frac{\partial f}{\partial x} = 0$: $f - y'\frac{\partial f}{\partial y'} = \mathrm{const}$: show this by considering $\frac{df}{dx}$

To (possibly) determine the nature of a stationary $y(x)$, consider the second variation (coefficient of ϵ^2):

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$$\delta^2 F = \frac{1}{2}\int_a^b Q\eta^2 + P\eta'^2 dx \quad \text{where} \quad Q(x) = \frac{\partial^2 f}{\partial y^2} - \frac{d}{dx}\frac{\partial^2 f}{\partial y \partial y'}, \quad P(x) = \frac{\partial^2 f}{\partial y'^2}$$

where $P(x)$ and $Q(x)$ are evaluated for the stationary $y(x)$. If this is positive for all suitable non-zero $\eta(x)$, then $y(x)$ is a local minimum (similarly negative for maximum, and mixed for saddle). Have various partial results:

- (Necessary condition; Legendre condition) if $y(x)$ is a local minimum then $P(x) \geq 0$ for all x
- (Sufficient condition) $P(x) > 0$ and $Q(x) > 0$ for all x: is *sufficient* for $y(x)$ to be a local minimum. Can relax this a little for P and Q to have some zeros, so long as integral for $\delta^2 F$ is always positive for any suitable non-zero $\eta(x)$.
- (Associated eigenvalue problem) define $\mathcal{L}$ as $\mathcal{L}\eta = -(P\eta')' + Q\eta$, and hence $\delta^2 F = \frac{1}{2}\int \eta[\mathcal{L}\eta]dx$. The signs of the eigenvalues of $\mathcal{L}\eta = \lambda\eta$ (with $\eta = 0$ at the ends) can be used to analogously to the Hessian (all positive for a minimum).
- (Jacobi condition) If there is a $u(x)$ on $[a, b]$ satisfying $\mathcal{L}u = 0$ and $u(x) \neq 0$ for any x, then $y(x)$ is a minimiser. This comes from completing the square in the integrand, in essence:

$$\int P\eta'^2 + Q\eta^2 dx = \int P\left(\eta' - \frac{u'}{u}\eta\right)^2 + \frac{\mathcal{L}u}{u}dx.$$

3 Convexity and Legendre transforms

A set S **is convex** if $(1-t)\mathbf{x} + t\mathbf{y} \in S$ for all $\mathbf{x}, \mathbf{y} \in S$ and for $0 \leq t \leq 1$.

A function f **is convex** if its domain is a convex set and $f\left((1-t)\mathbf{x} + t\mathbf{y}\right) \leq (1-t)f(\mathbf{x}) + tf(\mathbf{y})$. Equivalently

- (i) $f(\mathbf{y}) \geq f(\mathbf{x}) + (\mathbf{y} - \mathbf{x}).\nabla f(\mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in S$
- (ii) $(\mathbf{y} - \mathbf{x}).(\nabla f(\mathbf{y}) - \nabla f(\mathbf{x})) \geq 0$ for all $\mathbf{x}, \mathbf{y} \in S$
- (iii) The Hessian has all $\lambda \geq 0$ throughout the domain

The **Legendre transform** f^* is given by $f^*(\mathbf{p}) = \sup_{\mathbf{x}}\left[\mathbf{p}.\mathbf{x} - f(\mathbf{x})\right]$. The function f^* is always convex, and $f^{**} = f$ if f is convex.

4 Formalisms of Dynamics (Lagrangian, Hamiltonian, and Noether)

Given the kinetic energy $T(\mathbf{q}, \dot{\mathbf{q}}, t)$ and the potential energy $V(\mathbf{q}, \dot{\mathbf{q}}, t)$, the **Lagrangian** is $L(\mathbf{q}, \dot{\mathbf{q}}, t) = T - V$. The dynamics are given by minimising the action $\int L\, dt$.

A special case of **Noether's theorem** gives that if $\mathbf{X}(t, s)$ is a symmetry of $\mathbf{x}$ for some f, then $\frac{\partial f}{\partial \dot{x}_i}\frac{dX_i}{ds}\Big|_{s=0}$ is constant (with summation convention).

The **Hamiltonian** is given by taking the Legendre transform of the Lagrangian with respect to $\dot{\mathbf{q}}$ (generalised velocity) to introduce $\mathbf{p}$, the generalised momentum. Assuming convexity of L with respect to $\dot{\mathbf{q}}$, the Hamiltonian can be expressed as $H(\mathbf{q}, \mathbf{p}, t) = \mathbf{p}.\dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}, t)$ with $\dot{\mathbf{q}}$ such that $\mathbf{p} = \frac{dL}{d\dot{\mathbf{q}}}$. The consequence of minimising action of is **Hamilton's equations**:

$$\dot{\mathbf{q}}_i = \frac{\partial H}{\partial \mathbf{p}_i}, \quad \dot{\mathbf{p}}_i = -\frac{\partial H}{\partial \mathbf{q}_i}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

If the system does not depend explicitly on time, H is constant.