Example Sheet 3

1. The Ricci identity for a vector field V^μ is $\nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu = R^\mu{}_{\nu\alpha\beta} V^\nu$.

(i) Deduce the corresponding result for a covector field W_μ by considering $\nabla_\alpha \nabla_\beta (V^\mu W_\mu) - \nabla_\beta \nabla_\alpha (V^\mu W_\mu)$. Is there an easier way to find this result?

(ii) Given two vector fields U^μ and V^μ, evaluate $\nabla_\alpha \nabla_\beta (U^\mu V^\nu) - \nabla_\beta \nabla_\alpha (U^\mu V^\nu)$. Deduce that, for an arbitrary tensor field $T^{\mu\nu}$,

$$\nabla_\alpha \nabla_\beta T^{\mu\nu} - \nabla_\beta \nabla_\alpha T^{\mu\nu} = R^\mu{}_{\sigma\alpha\beta} T^{\sigma\nu} + R^\nu{}_{\sigma\alpha\beta} T^{\mu\sigma}.$$

Hence show that $\nabla_\alpha \nabla_\beta T^{\alpha\beta} = \nabla_\beta \nabla_\alpha T^{\alpha\beta}$, for any tensor field $T^{\alpha\beta}$.

2. Show that, if a vector S^α is parallelly transported along an affinely parametrized geodesic $x^\alpha(\lambda)$ with tangent vector T^α, then $g_{\alpha\beta}\, S^\alpha T^\beta$ is constant along the curve.

Consider the parallel transport of a vector $S^\alpha = (S^\theta, S^\phi)$ around a closed path on the unit 2-sphere, with θ, ϕ the usual polar coordinates. The path consists of the following four segments:
(i) $\theta = \frac{1}{2}\pi,\ \phi_0 \geq \phi \geq 0$, (ii) $\frac{1}{2}\pi \geq \theta \geq \theta_0,\ \phi = 0$, (iii) $\theta = \theta_0,\ 0 \leq \phi \leq \phi_0$, (iv) $\theta_0 \leq \theta \leq \frac{1}{2}\pi,\ \phi = \phi_0$,
and $S^\alpha = (1, 0)$ at the starting point, $\theta = \frac{1}{2}\pi$, $\phi = \phi_0$ (on the equator).

(a) Sketch a picture in the case $\theta_0 = 0$ (so the path is a spherical triangle with one vertex at the North pole) using the result of the first paragraph (no further calculation required) and hence show that the angle between the initial and final vectors S^α is proportional to the area enclosed by the path.

(b) Verify that for $0 < \theta_0 < \frac{1}{2}\pi$ the parallel transport equation has the following solutions for S^α on each segment:
(i) $(1, 0)$, (ii) $(1, 0)$, (iii) $(\cos(c_0\phi), -\sin(c_0\phi)/\sin\theta_0)$, (iv) $(\cos(c_0\phi_0), -\sin(c_0\phi_0)/\sin\theta)$, where $c_0 = \cos\theta_0$.
Write down S^α at the end point of the path and check that, when $\theta_0 \to 0$, this agrees with the result in part (a).
[The non-zero connection components on the 2-sphere are $\Gamma_\phi{}^\theta{}_\phi = -\sin\theta\,\cos\theta$ and $\Gamma_\theta{}^\phi{}_\phi = \Gamma_\phi{}^\phi{}_\theta = \cot\theta$.]

3. Show, by considering its symmetries, that the Riemann curvature tensor for a metric on a 2-dimensional manifold has only one independent component. Show further that for such a metric

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2} R (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) \,.$$

Verify this result using the connection components for 2-dimensional de Sitter spacetime (obtained in question 9 on Example Sheet 1).

4. Let ϕ be a scalar field in curved spacetime such that

$$\nabla_\alpha \nabla_\beta\, \phi \;=\; R_{\alpha\beta} \,,$$

where $R_{\alpha\beta}$ is the Ricci tensor. Show that

$$\nabla_\alpha (\nabla_\beta \nabla^\beta \phi) \;=\; -2 R_{\alpha\beta} \nabla^\beta \phi$$

and hence deduce that $\nabla_\alpha \phi \nabla^\alpha \phi + R$ is constant.
[*Hint: use the Ricci identity and the contracted Bianchi identity,* $\nabla^\beta R_{\alpha\beta} = \frac{1}{2} \nabla_\alpha R$.]

5. The Maxwell tensor $F_{\alpha\beta}$ for the electromagnetic field in curved spacetime is given in terms of a vector potential A_α by $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$. Show that this implies $\nabla_{[\gamma} F_{\alpha\beta]} = 0$. Show further that if $\nabla_\beta F^{\alpha\beta} = 0$, then the energy momentum tensor

$$T^{\alpha\beta} \;=\; F^{\alpha}{}_{\gamma} F^{\beta\gamma} - \frac{1}{4} g^{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta}\,,$$

is conserved, i.e. $\nabla_\beta T^{\alpha\beta} = 0$.

6. Let ξ_α be a Killing covector field, satisfying $\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$ (see question 5 on Example Sheet 2). Use the Ricci identity and $R^{\alpha}{}_{[\beta\gamma\delta]} = 0$ to show that

$$\xi_{\alpha;\beta\gamma} = -R^{\delta}{}_{\gamma\alpha\beta}\,\xi_\delta\,.$$

In the case of Minkowski space, integrate this equation twice and deduce that there are 10 independent Killing vectors.

7. Starting from the formula for the Levi-Civita connection, find an expression for the Riemann tensor $R_{\alpha\beta\gamma\delta}$ in local inertial coordinates. Hence prove that

$$R_{\alpha\beta\gamma\delta} \;=\; R_{\gamma\delta\alpha\beta}\,.$$

8. Consider Newtonian spacetime with (Cartesian) coordinates $x^\alpha = (t, x^i)$. Show that the motion of a freely-falling particle can be described by a curve $x^\alpha(\lambda)$, where

$$\frac{d^2 t}{d\lambda^2} = 0\,, \qquad \frac{d^2 x^i}{d\lambda^2} + \frac{\partial \Phi}{\partial x^i}\Big(\frac{dt}{d\lambda}\Big)^2 = 0\,,$$

for a suitable parameter λ and Newtonian gravitational potential $\Phi(x^i)$. Regarding this as a geodesic equation, read off the Newtonian connection components and deduce that the corresponding curvature is given by

$$R^{i}{}_{0j0} = -R^{i}{}_{00j} = \frac{\partial^2 \Phi}{\partial x^i \partial x^j}\,, \qquad R^{\alpha}{}_{\beta\gamma\delta} = 0 \quad \text{otherwise}\,.$$

Can this Newtonian connection and curvature arise from a metric? [*Hint: For the standard Levi-Civita connection, what symmetries does the Riemann tensor possess?*]

9. The Lie derivative $(\mathcal{L}_\xi V)^\alpha$ of a vector field V^α with respect to a vector field ξ^α (assumed to be timelike) is defined by the following conditions:
(i) If $\{x^\alpha\}$ is a coordinate system in which $\xi^\alpha = (1,0,0,0)$, then

$$(\mathcal{L}_\xi V)^\alpha \;=\; \frac{\partial V^\alpha}{\partial x^0} \;=\; \xi^\beta \frac{\partial V^\alpha}{\partial x^\beta}\,,$$

and (ii) $(\mathcal{L}_\xi V)^\alpha$ transforms as a vector. Show that, in a general coordinate system,

$$(\mathcal{L}_\xi V)^\alpha \;=\; \xi^\beta \nabla_\beta V^\alpha \;-\; V^\beta \nabla_\beta \xi^\alpha\,.$$

Suppose, in addition, that the Lie derivative $\mathcal{L}_\xi \phi$ of a scalar field ϕ with respect to a vector field ξ^α is defined in a general coordinate system $\{x^\alpha\}$ by

$$\mathcal{L}_\xi \phi \;=\; \xi^\alpha \frac{\partial \phi}{\partial x^\alpha}$$

and that the Lie derivative obeys the usual Leibniz rule when applied to a tensor product. Find the Lie derivative $(\mathcal{L}_\xi U)_\alpha$ of a covector field U_α.

Write down an expression for the Lie derivative with respect to ξ^α of a $(0,2)$ tensor $T_{\alpha\beta}$ and show that the condition for ξ^α to be a Killing vector field (as in question 6 above) is $(\mathcal{L}_\xi g)_{\alpha\beta} = 0$, where $g_{\alpha\beta}$ is the metric tensor.

10. (i) Let M be an invertible matrix. Show that under a small change δM, the corresponding change in the determinant is, to first order, $\delta(\det M) = (\det M)\,\mathrm{tr}(M^{-1}\delta M)$.

[*Hint: if the entries of a matrix A are small then, to first order,* $\det(I+A) = 1 + \mathrm{tr}A$*, where I is the identity matrix.*]

(ii) Let $g_{\alpha\beta}$ be a metric with Lorentzian signature and let $g = \det(g_{\alpha\beta})$. Use the result in (i) to show that

$$\Gamma^{\ \beta}_{\alpha\ \beta} = \frac{1}{2g}\frac{\partial g}{\partial x^\alpha} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\sqrt{-g}\,,$$

where $\Gamma^{\ \gamma}_{\alpha\ \beta}$ is the Levi-Civita connection. (Note that $g < 0$ for a metric with Lorentzian signature.)

(iii) A *tensor density of weight* q is defined to be a quantity that transforms as a tensor under a change in coordinates from $\{x^\mu\}$ to $\{\tilde{x}^\alpha\}$ but with an additional factor of Δ^q, where $\Delta = \det(\partial x^\mu/\partial\tilde{x}^\alpha)$, the Jacobian. Show that g transforms as a scalar density of weight 2.

(iv) For ψ a scalar density of weight q, the covariant derivative is defined by

$$\nabla_\alpha\psi = \frac{\partial\psi}{\partial x^\alpha} - q\,\Gamma^{\ \beta}_{\alpha\ \beta}\psi\,.$$

Show that $\nabla_\alpha\psi$ is a covector density of weight q.

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