3. Interlude: Dimensional Analysis

The essence of dimensional analysis is very simple: if you are asked how hot it is outside, the answer is never "2 o'clock". You've got to make sure that the units agree. Quantities which come with units are said to have *dimensions*. In contrast, pure numbers such as 2 or π are said to be *dimensionless*.

In all the examples that we met in the previous section, the units are hiding within the variables. Nonetheless, it's worth our effort to dig them out. In most situations, it is useful to identify three fundamental dimensions: length L, mass M and time T. The dimensions of all other quantities should be expressible in terms of these. We will denote the dimension of a quantity Y as $[Y]$. Some basic examples include,

$$\begin{aligned} [\text{Area}] &= L^2 \\ [\text{Speed}] &= LT^{-1} \\ [\text{Acceleration}] &= LT^{-2} \\ [\text{Force}] &= MLT^{-2} \\ [\text{Energy}] &= ML^2T^{-2} \end{aligned}$$

The first three should be obvious. You can quickly derive the last two by thinking of your favourite equation and insisting that the dimensions on both sides are consistent. For example, $F = ma$ immediately gives the dimensions $[F]$, while $E = \frac{1}{2}mv^2$ will give you the dimensions $[E]$. This same technique can be used to determine the dimensions of any constants that appear in equations. For example, Newton's gravitational constant appears in the formula $F = -GMm/r^2$. Matching dimensions on both sides tells us that

$$[G] = M^{-1}L^3T^{-2}$$

You shouldn't be too dogmatic in insisting that there are exactly three dimensions of length, mass and time. In some problems, it will be useful to introduce further dimensions such as temperature or electric charge. For yet other problems, it could be useful to distinguish between distances in the x-direction and distances in the z-direction. For example, if you're a sailor, you would be foolish to think of vertical distances in the same way as horizontal distances. Your life is very different if you mistakenly travel 10 fathoms (i.e. vertically) instead of 10 nautical miles (i.e. horizontally) and it's useful to introduce different units to reflect this.

Conversely, when dealing with matters in fundamental physics, we often reduce the number of dimensional quantities. As we will see in Section 7, in situations where special relativity is important, time and space sit on the same footing and can be measured in the same unit, with the speed of light providing a conversion factor between the two. (We'll have more to say on this in Section 7.3.3). Similarly, in statistical mechanics, Boltzmann's constant provides a conversion factor between temperature and energy.

Scaling: Bridgman's Theorem

Any equation that we derive must be dimensionally consistent. This simple observation can be a surprisingly powerful tool. Firstly, it provides a way to quickly check whether an answer has a hope of being correct. (And can be used to spot where a mistake appeared in a calculation). Moreover, there are certain problems that can be answered using dimensional analysis alone, allowing you to avoid calculations all together. Let's look at this in more detail.

We start by noting that dimensionful quantities such as length can only appear in equations as powers, L^α for some α. We can never have more complicated functions. One simple way to see this is to Taylor expand. For example, the exponential function has the Taylor expansion

$$e^x = 1 + x + \frac{x^2}{2} + \ldots$$

The right-hand side contains all powers of x and only makes sense if x is a dimensionless quantity: we can never have e^L appearing in an exponent otherwise we'd be adding a length to an area to a volume and so on. A similar statement holds for $\sin x$ and $\log x$, for your favourite and least favourite functions. In all cases, the argument must be dimensionless unless the function is simply of the form x^α. (If your favourite function doesn't have a Taylor expansion around $x = 0$, simply expand around a different point to reach the same conclusion).

Suppose that we want to compute some quantity Y. This must have dimension

$$[Y] = M^\alpha L^\beta T^\gamma$$

for some α, β and γ. (There is, in general, no need for these to be integers although they are typically rational). We usually want to determine Y in terms of various other quantities in the game – call them X_i, with $i = 1, \ldots n$. These too will have certain dimensions. We'll focus on just three of them, X_1, X_2 and X_3. We'll assume that these three quantities are "dimensionally independent", meaning that by taking suitable combinations of X_1, X_2 and X_3, we can build quantities with dimension of length, mass and time. Then we must be able to express Y as

$$Y = C\, X_1^{a_1} X_2^{a_2} X_3^{a_3}$$

for some a_1, a_2 and a_3 such that

$$[X_1^{a_1}]\,[X_2^{a_2}]\,[X_3^{a_3}] = M^\alpha L^\beta T^\gamma$$

which is simply the requirement that the dimensions agree on both sides. All the difficulty of the problem has been swept into determining C which, by necessity, is dimensionless. In principle, C can depend on all the X_i. However, since C is dimensionless, it can only depend on combinations of X_i which are also dimensionless. And this will often greatly restrict the form that the answer can take.

An Example: The Pendulum

The above discussion is a little abstract. Let's throw some light on it with a simple example. We will consider a pendulum. We already discussed the pendulum earlier in (2.9). It has equation of motion

$$\ddot{\theta} = -\frac{g}{l} \sin\theta$$

We'd like to know the period, T. This plays the role of the quantity we called Y above: clearly, it has dimension of time. (Although we've picked a slightly annoying choice of notation because we have the equation $[T] = T$. Hopefully it won't cause too much confusion).

What are the variables X_i that the period can depend upon? There are four of them: the strength of gravity g, the mass of the pendulum m, the length of the pendulum l and the initial starting angle θ_0. The dimension of m and l are obviously mass and length respectively; the dimension of acceleration is $[g] = LT^{-2}$ while the initial angle is necessarily dimensionless $[\theta_0] = 0$. (This follows from its periodicity, $\theta = \theta + 2\pi$, because 2π is dimensionless; alternatively it follows from the fact that it sits as the argument of a sin function). The only dimensionless combination that we can form is θ_0 itself. We can therefore write

$$T = C(\theta_0)\, g^{a_1} m^{a_2} l^{a_3}$$

where, on dimensional grounds, we must have

$$[T] = T = [g^{a_1}]\,[m^{a_2}]\,[l^{a_3}] = M^{a_2} L^{a_1+a_3} T^{-2a_1}$$

The unique solution is $a_2 = 0$ and $a_1 = -a_3 = -\frac{1}{2}$. We learn immediately that

$$T = C(\theta_0)\sqrt{\frac{l}{g}} \tag{3.1}$$

This agrees with the result (2.10) that we got the hard way by solving the equation of motion. Of course we haven't solved the problem completely; by using dimensional analysis there's no way to figure out the function $C(\theta_0)$ which is given by the elliptic integral in (2.10).

Nonetheless, there's important information contained in the form (3.1). For example, it tells us that the mass of the pendulum doesn't affect the period. Moreover, suppose you are given two pendulums, with lengths l_1 and l_2. You release them from the same starting angle and want to know how much faster the first pendulum swings compared to the second. For these kinds of comparative questions, the unknown function $C(\theta_0)$ drops out, and we can just immediately write down the result:

$$\frac{T_1}{T_2} = \sqrt{\frac{l_1}{l_2}}$$

Whenever we are interested only in how things scale with some quantity, it is conventional to use the symbol $\sim$. (We could also use the proportional symbol $\propto$ but it looks a little too much like the Greek letter α). So equation (3.1) would be written as

$$T \sim \sqrt{\frac{l}{g}}$$

In fact we already used this notation a number of times in the last Section.

The Importance of Dimensionless Quantities

The power of dimensional analysis really depends on how many dimensionless quantities we can construct from the variables at hand. If we can construct r dimensionless variables, then the unknown dimensionless quantity C is a function of r variables. In problems where $r = 0$ and there are no dimensionless combinations of variables, then C is just a number.

It is a simple matter to count the number of dimensionless parameters in a given problem. If we have n independent variables X_i in a problem that requires k independent dimensions then we will be able to form $r = n - k$ dimensionless combinations. (In our discussion above, we had $k = 3$ corresponding to mass, length and time). This intuitive result goes by the grand name of the *Buckingham* Π *theorem*. It can be proved formally by setting up a system of linear equations and invoking the rank-nullity theorem of linear algebra. Finally, the dimensionless combinations that you can make in a given problem are not unique: if x and y are both dimensionless, then so are xy and x^2y and $x+y$ and, indeed, any function that you want to make out of these two variables.

There are other reasons to be interested in dimensionless quantities. The first is practical: identifying dimensionless quantities at an early stage in a calculation will save you ink! In a calculation that contains lots of variables, you'll often find the same dimensionless combinations of variables appearing at every stage. In particular – as we've already seen – it is only dimensionless combinations that can appear as the arguments of functions. Often, identifying these combinations at an early stage — and perhaps even giving them a name of their own — will speed up the computation and help in avoiding errors.

For example, if we look back to the problem of the 3d projectile with linear friction, with equation of motion (2.29), we see that the dimensionless combination $\gamma t/m$ appears over and over in all steps of the calculation. In this case, it wasn't too annoying to keep writing $\gamma t/m$. But if you find yourself doing a calculation where the combination $e^2 m_e/2\pi\epsilon_0\hbar^2 r$ appears three times on every line, then it's a good idea to come up with a new name for this object.

The second reason to be interested in dimensionless quantities is because the answer to a calculation often simplifies in certain regimes. Perhaps this is the regime of long times, or short distances, or high speeds, or some such thing. But only dimensionless numbers can be big. For a dimensionless quantity x, we can write $x \gg 1$. But it makes no sense to write $Y \gg 1$ if Y is not dimensionless: a dimensionful quantity must always be big or small relative to something else.

We already discussed this issue in the case of the projectile (2.29), where we saw that long times necessarily meant $t\gamma/m \gg 1$. This is also the reason that we needed to introduce a dimensionless quantity, the Reynolds number (2.25), to decide which systems suffer linear friction vs quadratic friction.

Another Example: The Atomic Bomb

In the 1950s, the fluid dynamicist G.I. Taylor applied dimensional analysis to photographs of an atomic explosion. As you can see in the example below, these photographs happily came with both a time scale and distance scale, allowing you to trace the radius of the shock front $R(t)$ as a function of time after the explosion. To the annoyance of the US government, Taylor was able to use these time and distance scales to get a good estimate of the energy released in the explosion. At the time this was classified information.

For most explosions, the dynamics of the shock front depends on the pressure of the outside air. Taylor's insight was to realise that in an explosion as powerful as an atomic bomb, the air pressure is completely negligible. However, the density of air, ρ, is important. Taylor identified the following relevant variables

$$\begin{aligned} \text{Air density} \quad & [\rho] = ML^{-3} \\ \text{Shock Front Radius} \quad & [R] = L \\ \text{Time from Explosion} \quad & [t] = T \end{aligned}$$

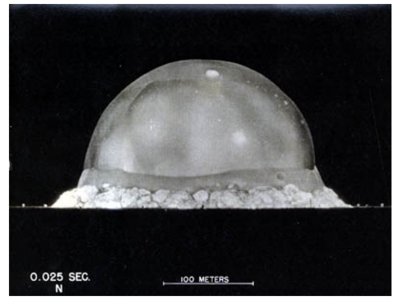

Figure 10:

There are no dimensionless quantities that we can build from these. Since the energy released in the explosion has dimension $[E] = ML^2T^{-2}$, on dimensional grounds we must have

$$E = C\,\frac{\rho R^5}{t^2}$$

where C is an unknown constant. Of course, without knowing C this would seem to be useless. In Taylor's case, a few further supplementary calculations allowed him to estimate C.

In general, there's a good rule of thumb if you want to figure out unknown constants such as C: once you've figured out how many factors of 2π they contain, what's left is almost always a number that's close to one. With a little bit of experience, it's usually possible to guess the factors of 2π as well since they usually arise for some geometric reason. All of which means that dimensional analysis is, perhaps, even more unreasonable useful than we might have originally hoped. Here, for example, is Einstein himself weighing in on the issue:

> "*...for why should not a numerical factor like* $(12\pi)^3$ *appear in a mathematical-physical deduction? But without doubt, such cases are rarities!*

A Last Example: Rowing

Another, classic demonstration of the power of dimensional analysis is in understanding how the speed of a rowing boat depends on the number of rowers[2].

[2] This analysis was first by T. McMahon in the paper "*Rowing: A similarity analysis*", Science 173:349 (1971)

The boat experiences quadratic friction, proportional to the submerged cross-sectional area A of the boat.

$$F_{\rm drag} \sim v^2 A$$

(On dimensional grounds, we actually have $F \sim \rho v^2 A$, where ρ is the density of water, but this will not be important for our story). The *power* needed to overcome the drag is therefore

$$P = F_{\rm drag} v \sim v^3 A$$

By Archimedes' law, the displaced volume increases linearly with the number of rowers, N. This means that the submerged volume $V \sim N$ so the submerged area scales as $A \sim N^{2/3}$. (We are assuming here that the mass of the boat is negligible compared to the mass of the rowers). Meanwhile, if we further assume that the power supplied by each rower is the same, we have $P \sim N$. Putting all this together, we have $P \sim N \sim v^3 N^{2/3}$. Rearranging, we learn that the velocity increases with the number of rowers as

$$v \sim N^{1/9}$$

This simple result actually agrees pretty well with Olympic rowing times.

Dimensional Constants of Nature

The laws of physics provide us with three fundamental constants of Nature. We have already met $G \approx 6.7 \times 10^{-11}\ m^3 \text{Kg}^{-1} s^{-2}$ which appears in both Newton's law of gravity as well as the more refined theory of gravity due to Einstein known as general relativity. The other two fundamental constants are the speed of light, $c \approx 3 \times 10^8\ ms^{-1}$, which characterises the relationship between space and time in special relativity, and Planck's constant $\hbar \approx 10^{-34}\ Js$ which determines when quantum effects become important.

These constants have dimensions

$$[G] = M^{-1}L^3T^{-2} \quad , \quad [c] = LT^{-1} \quad , \quad [\hbar] = ML^2T^{-1}$$

From these three constants, we can construct a characteristic length scale, known as the *Planck length* l_p

$$l_p = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-35} m$$

This is the distance at which gravity, quantum mechanics and the structure of space-time all become important. All indications are that this is the shortest distance scale possible; at distances shorter than l_p, space itself is likely to have no meaning. Similarly, we can define the Planck time, $t_p = l_p/c$, the Planck mass $m_p = \sqrt{\hbar c/G}$ and the Planck energy,

$$E_p = \frac{\hbar c}{l_p} \approx 10^{19}\ GeV$$

where $1\ GeV \approx 10^{-10}\ J$ is a measure of energy used in particle physics. If we want to explore aspects of quantum gravity in experiments on Earth, we will need to build particle colliders capable of reaching Planck energies. This is a long way off: the LHC operates at energies around $10^4\ GeV$.